INTERNATIONAL CONFERENCE

DAYS ON DIFFRACTION 2020

ABSTRACTS

May 25 – 29, 2020

St. Petersburg

ORGANIZING COMMITTEE

V. M. Babich /Chair/, M. I. Belishev /Vice-chairman/,
A. S. Kirpichnikova /Secretary/,
I. V. Andronov, L. I. Goray, A. Ya. Kazakov,
N. Ya. Kirpichnikova, A. P. Kiselev, M. A. Lyalinov,
V. S. Mikhaylov, O. V. Motygin, M. V. Perel,
V. P. Smyshlyaev, R. Stone, N. Zhu

Conference e-mail: `diffraction2020@gmail.com`

Web site: `http://www.pdmi.ras.ru/~dd/`

The conference is organized and sponsored by

St. Petersburg Department of V.A. Steklov Institute of Mathematics

St. Petersburg State University

The Euler International Mathematical Institute

IEEE Russia (Northwest) Section AP/ED/MTT Joint Chapter

GOVERNMENT OF THE RUSSIAN FEDERATION

SIMONS FOUNDATION

FOREWORD

“Days on Diffraction” is an annual conference taking place in May–June in St. Petersburg since 1968. The present event is organized by St. Petersburg Department of the Steklov Mathematical Institute, St. Petersburg State University, and the Euler International Mathematical Institute.

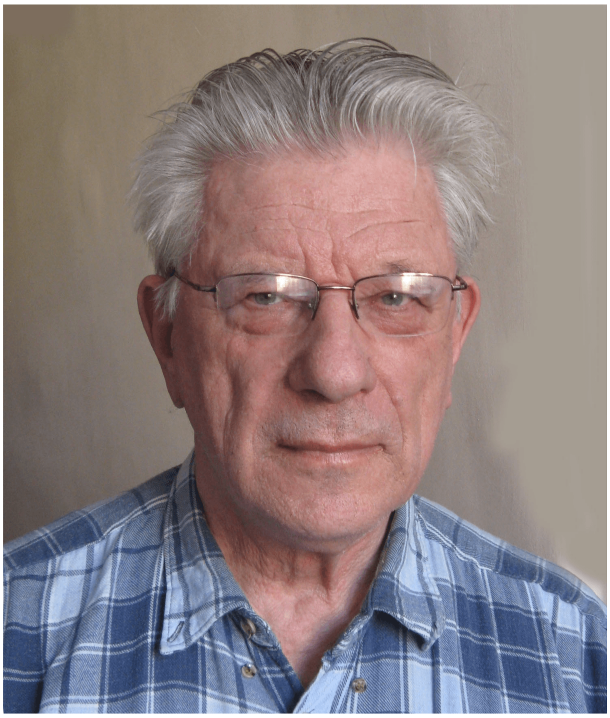
The conference is supported by a grant from the Government of the Russian Federation, agreement № 075-15-2019-1620, and by Simons Foundation (via PDMI RAS, grant № 507309).

The abstracts of 57 talks, presented during 5 days of the conference, form the contents of this booklet. The author index is located on the last page.

Full-length texts of selected talks will be published in the Conference Proceedings. Format file and instructions can be found at `http://www.pdmi.ras.ru/~dd/proceedings.php`. The final judgement on accepting the paper for the Proceedings will be made by editorial board after peer reviewing.

Organizing Committee

© ‘Days on Diffraction’, PDMI, 2020



90 years to V. M. Babich

The Organizing Committee of "Days on Diffraction 2020" congratulates Vassily Mikhailovich Babich on the occasion of his 90th birthday, celebrated by the diffraction theory community on 13 June 2020. V. M. is a bright member of the Leningrad/St. Petersburg school of mathematical physics known for his outstanding contributions to the mathematical theory of diffraction and wave propagation and for his personal influence. In 1954 V. M. began his teaching career at his alma mater (the Department of Mathematical Physics of the Leningrad State University). He also heads the Laboratory for Mathematical Methods in Geophysics at the St. Petersburg Department of the Steklov Mathematical Institute (PDMI) since 1967. The undisputed reputation of his weekly seminar at the PDMI is based on the very high level of talks.

For his work on applications of the ray method to propagation of seismic waves, V. M. Babich was awarded State prize of the Soviet Union. In 1998 he was awarded V. A. Fock prize for his results developing asymptotic methods in the diffraction theory. The level of his studies is characterised by the fact that his paper on waves in anisotropic media was reprinted in a leading international geophysical journal 33 years after its original publication in Russian. This year, Vassily Mikhailovich chairs his 53rd "Days on Diffraction", and his name is inseparably linked with this annual conference. The Organizing committee expresses its sincere gratitude and deep respect to Vassily Mikhailovich and wishes him sound health and new achievements in his research.

List of talks

The Toffoli gate in multihelical optical fibres

Alexeyev C.N., **Barshak E.V.**, Vikulin D.V., Lapin B.P., Yavorsky M.A.
V. I. Vernadsky Crimean Federal University, Pr. Vernadsky 4, Simferopol, 295007, Russia
e-mail: barshakev@cfuv.ru

Nowadays, optical vortices (OVs) are becoming progressively more important as beams which carrier of the orbital angular momentum (OAM) [1, 2]. Indeed, advanced research in the information, telecommunications and quantum technologies is associated with the use as a carrier of information not only spin (associated with polarization), but also the OAM of a photon. The information encoding in the OAM-associated degrees of freedom of light has the advantages of extremely high data-carrying capacity of a communication channel, as compared to the standard techniques, and provides a high level of the resistance to eavesdropping.

Since orthogonal states of OVs characterized by different OAM values form a multidimensional space, this question turns out to be closely related to the possibility of modeling quantum calculations by means of classical optical fields [3]. Obviously, comprehensive use of information OAM-potential for modeling quantum calculations requires having a mechanism for carrying out basic logical operation, or gates [4].

In this work the effect of topological-charge controlling of the output optical vortex via a changing of sign of the circular polarization and a radial number of the input beam in twisted multihelical optical fibres is discussed. Based on this effect the way to create a universal logical gate Toffoli is described. Gate Toffoli is a 3-qbit-gate, which is also known as the “controlled-controlled-not” gate (see Fig. 1). We suppose that this type of all-fiber OAM-controlling would be useful in such OV-based applications as classical and quantum information encryption and simulation of quantum computing.

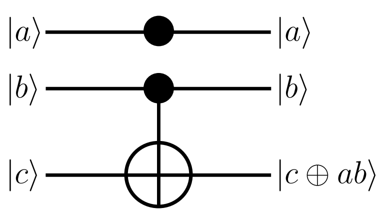


Fig. 1: Circuit representation of Toffoli gate (CCNOT-gate).

This work was financially supported by the Russian Ministry of Education and Science, Megagrant project № 075-15-2019-1934 and also by V. I. Vernadsky Crimean Federal University, project ВГ02/2020.

References

[1] A. Yao, M. Padgett, Orbital angular momentum: origins, behavior and applications, *Advances in Optics and Photonics*, **3**, 161–204 (2011).

[2] A. E. Willner, H. Huang, Y. Yan, et al., Optical communications using orbital angular momentum beams, *Advances in Optics and Photonics*, **7**, 66–106 (2015).

[3] R. J. C. Spreeuw, Classical wave-optics analogy of quantum-information processing, *Physical Review A*, **63**, 062302 (2001).

[4] M. A. Yavorsky, E. B. Barshak, D. V. Vikulin, C. N. Alexeyev, Spin-dependent OAM flipping in multihelical optical fibres, *Journal of Optics*, **20**, 115601 (2018).

Semi-classical asymptotics of spectral bands for rotating dimers

Anikin A.Yu.
RAS Ishlinsky Institute for Problems in Mechanics, pr. Vernadskogo, 101 bld., Moscow, 1119526, Russia
e-mail: anikin83@inbox.ru

We deal with the Schrödinger operator $\widehat{H} = -h^2\Delta/2 + V(x)$ in the semi-classical limit describing the interacting pair of particles moving in the two-dimensional periodic trigonally symmetric potential field. Namely, we assume that $V(x) = U_0(x_1, x_2) + U_0(x_3, x_4) + U_1(x)$, where U_0 is periodic on a 2D lattice and has a trigonal symmetry, and $U_1(x)$ describes the interaction between two particles. We discuss the asymptotics as $h \to 0$ for the ground state spectral band widths as well as the dispersion relations between energy and quasi-momentum and the form of Bloch functions. Studying this sort of quantum systems called the 'rotating dimers' was proposed by M. I. Katsnelson and motivated by the physics of graphene.

The work was supported by the Russian Foundation for Basic Research (grant № 18-31-00273).

Uniform asymptotic solution in the form of an Airy function for semiclassical bound states in one-dimensional problems

Anikin A.Yu., Dobrokhotov S.Yu., Nazaikinskii V.E., **Tsvetkova A.V.**
Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences, Moscow, Russia
e-mail: anikin83@inbox.ru, dobr@ipmnet.ru, nazay@ipmnet.ru, annatsvetkova25@gmail.com

Let h be a small positive parameter and $\hat{p} = -ih\frac{d}{dx}$. On the real axis $\mathbb{R}$ with coordinate x, we consider the family of h-pseudodifferential operators [1]

$$\hat{H}(E) = H(x, \hat{p}, E, h) \tag{1}$$

acting on m-dimensional vector functions $\Psi(x) = (\psi_1(x), \psi_2(x), ..., \psi_m(x))$ with a smooth $m \times m$ matrix-valued symbol $H(x, p, E, h) = H_0(x, p, E) + hH_1(x, p, E) + O(h^2)$.

We consider the spectral problem

$$\hat{H}(E)\Psi = 0, \quad \Psi \in L^2(\mathbb{R}; \mathbb{C}^m) \tag{2}$$

for operator pencil (1), where the unknowns are the number E and the vector function $\Psi(x)$.

We study the formal asymptotic solution (or quasimode) of the problem (2) as $h \to 0$. We obtain efficient formulas for the leading term of a set of quasimodes in terms of Airy function under certain conditions on the leading symbol H_0 of the operator pencil $\hat{H}(E)$.

We also discuss some examples. In particular, Shrödinger equation and its generalization can be considered as examples. We also present the results for the pseudodifferential equation describing water waves above an uneven bottom given by a function $D(x)$ depending only on x_1 (see [2]) and for the two-dimensional Dirac operator with a radially symmetric potential describing the quantum states in graphene located in a constant magnetic field (see [3]).

The work was supported by the Russian Science Foundation (project № 16-11-10282).

References

[1] V. P. Maslov, *Operational Methods*, Mir, Moscow, 1976.

[2] P. H. Le Blond, L. A. Mysak, *Waves in the Ocean*, Elsevier, Amsterdam, 1978.

[3] M. I. Katsnelson, *Graphene: Carbon in Two Dimensions*, Cambridge University Press, Cambridge, 2012.

On application of flag coordinates to extension of Lie–Poisson–Kirillov–Kostant structure from general linear group orbits

Babich M.V.
PDMI RAS, Fontanka 27, St. Petersburg, Russia
e-mail: mbabich@pdmi.ras.ru

The canonical method of the construction of the coadjoint orbit of the Lie group is the factorization of the right-invariant section of the group with respect to some subgroup. This subgroup consists of all the elements commuting with some fixed element from the dual of the corresponding Lie algebra. The subgroup form the kernel of the projection to the orbit.

I define the linear manifold that is canonically defined for any orbit. The subgroup acts on this manifold by invertible linear transformations. The set of these transformations is considered as manifold. The cotangent bundle of this manifold gives the desired extension of the coadjoint orbit.

All the considerations base on the concept of the flag coordinates on the orbit.

References

[1] M. V. Babich, On extensions of canonical symplectic structure from coadjoint orbit of complex general linear group, *Zap. Nauchn. Semin. POMI*, **487**, 28–39 (2019).

On an evolutionary dynamical system of the first order with boundary control

Belishev M.I.[1], **Simonov S.A.**[1,2]
[1]St. Petersburg Department of V. A. Steklov Institute of Mathematics of the Russian Academy of Sciences, 27 Fontanka, St. Petersburg, 191023, Russia
[2]St. Petersburg State University, 7-9 Universitetskaya nab., St. Petersburg, 199034, Russia
e-mail: belishev@pdmi.ras.ru, sergey.a.simonov@gmail.com

We consider an abstract evolutionary dynamic system of the first order (with respect to time) with boundary control, which is determined by a symmetric operator $L_0 : \mathscr{H} \to \mathscr{H}$. We show that it is controllable, if and only if L_0 has no maximal symmetric parts in $\mathscr{H}$. This work is carried out as part of the program to construct a new functional (so-called *wave*) model of symmetric operators.

Two-dimensional Dirac equation: singularities of semi-classical approximations

Bogaevsky I.A.
Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Leninskie Gory, Moscow GSP-1, 119991, Russia; Scientific Research Institute for System Analysis, Nakhimovskii prosp., 36/1, Moscow, 117218, Russia
e-mail: ibogaevsk@gmail.com

The two-dimensional massless Dirac equation describes the propagation of electrons and holes in graphene. We consider the Cauchy problem for this equation with an electromagnetic field and localized initial condition. The phase of its semi-classical approximation is described by a Lagrangian submanifold with singularities.

Generically for small times this Lagrangian submanifold can have only two normal forms up to canonical transformations. We give explicit formulas for them and investigate their singularities. Our two normal forms are realized by constant magnetic and electric fields.

The work is partially supported by RFBR and JSPS (research project № 19-51-50005).

Solution dynamic equations of plane deformation for nonlinear model of complex crystal lattice

Bulygin A.N., **Pavlov Yu.V.**
Institute for Problems in Mechanical Engineering of Russian Academy of Sciences, Saint Petersburg, Russia
e-mail: bulygin_an@mail.ru

The nonlinear model of deformation of crystalline media with complex lattice was proposed in [1, 2]. Deformation is described by vectors of acoustic $\mathbf{U}$ and optical $\mathbf{u}$ modes. For plane deformation, $\mathbf{U}$ and $\mathbf{u}$ can be found from a system of four coupled nonlinear equations [3, 4]:

$$\rho \frac{\partial^2 U_i}{\partial t^2} = \sigma_{ij,j}, \quad \sigma_{ij} = \lambda_{ijmn} e_{mn} - s_{ij}(1 - \cos u_s), \quad u_s = \frac{u_x + u_y}{b}, \quad e_{mn} = \frac{U_{m,n} + U_{n,m}}{2}, \tag{1}$$

$$\mu_0 \frac{\partial^2 u_i}{\partial t^2} = \chi_{ij,j} - \frac{1}{b}(p - s_{ij} e_{ij}) \sin u_s, \quad \chi_{ij} = \kappa_{ijmn} \varepsilon_{mn}, \quad \varepsilon_{mn} = \frac{u_{m,n} + u_{n,m}}{2}. \tag{2}$$

Here ρ, μ_0, b, p are density, reduced density, lattice cell size, half of the interatomic potential barrier, respectively; σ_{ij}, χ_{ij}, s_{ij} are tensors of stress, micro-stress, and nonlinear striction; λ_{ijmn}, κ_{ijmn} are elastic and microelastic tensors. In the nonlinear model (1), (2), the relative shifts of sublattices can be arbitrarily large, but the gradients of micro- and macro-shifts are assumed to be small.

The nonlinear model describes large deformations of the crystal cell, radical restructuring of the crystal medium under the influence of intense external effects, occurrence of defects of different kinds, phase transformations and other phenomena, which are realized in modern technologies of obtaining new materials with nanostructure, but are not described by the linear classical model.

We have found a common solution to the macrofield equations (1) for crystal media with cubic symmetry. The tensor σ_{ij} and vector $\mathbf{U}$ are expressed through the arbitrary function $Q(x, y, t)$, which is the dynamic analog of the Airy function. It is shown that $Q(x, y, t)$ satisfies a non-uniform bigarmonic equation. A general solution to this equation was found. It is represented through arbitrary analytic functions. Complex representation of the general solution of the macrofield equations (1) is given.

General solution to microfield equations (2) is found. The system of two coupled nonlinear equations (2) is reduced to two independent equations to find u_s and $u_m = (u_x - u_y)/b$. The function u_m is found as an arbitrary analytic function of a special argument. The function u_s is a solution to the dynamic double sine-Gordon equation, for which amplitude before $\sin u_s$ is an arbitrary harmonic function. Analysis of this equation has shown that under some limitations, it is reduced to equations that are well studied in the literature.

The obtained general solutions of equations of macro- and microfields allow us to set and solve specific problems for propagation of nonlinear waves in crystalline media of cubic symmetry.

References

[1] E. L. Aero, Microscale deformations in a two-dimensional lattice: structural transitions and bifurcations at critical shear, *Physics of the Solid State*, **42**, 1147–1153 (2000).

[2] E. L. Aero, Significantly nonlinear micromechanics of medium with variable periodic structure, *Uspekhi Mekhaniki*, **1**, 130–176 (2002).

[3] E. L. Aero, A. N. Bulygin, Yu. V. Pavlov, Nonlinear deformation model of crystal media allowing martensite transformations: solution of static equations, *Mechanics of Solids*, **53**, 623–632 (2018).

[4] E. L. Aero, A. N. Bulygin, Yu. V. Pavlov, Nonlinear model of deformation of crystalline media allowing for martensitic transformations: plane deformation, *Mechanics of Solids*, **54**, 797–806 (2019).

Determination of a wave field in a layered medium from boundary data

M.N. Demchenko
St. Petersburg Department of V. A. Steklov Institute of Mathematics of the Russian Academy of Sciences, Fontanka 27, St. Petersburg, Russia
e-mail: demchenko@pdmi.ras.ru

We deal with the Cauchy problem for the hyperbolic equation

$$\partial_t^2 u - \Delta u + Qu = 0 \tag{1}$$

for the function $u(x, y, t)$, $x, t \in \mathbb{R}$, $y > 0$, with data on the space-time boundary

$$u|_{y=0} = f, \quad \partial_y u|_{y=0} = g. \tag{2}$$

The problem in consideration is a mathematical formulation of the problem of continuation of a non-stationary wave field from the boundary. In contrast to the classical Cauchy problem with data on the set $\{t = t_0\}$, the problem (1), (2) is ill-posed. We provide an algorithm of local recovering of the solution u from local Cauchy data, i.e. from f, g given on a bounded set. We consider the special case when the coefficient Q has the following form

$$Q(x, y, t) = q(x) + p(y, t). \tag{3}$$

Under this assumption, equation (1) describes in particular a non-stationary wave process in a laterally or vertically inhomogeneous medium.

To find the solution u, we separate variables in equation (1), which is possible due to (3), and apply the eigenfunction expansion associated to the Schrödinger operator $-\partial_x^2 + q$ on the line. In general, this approach requires the Cauchy data on an unbounded set, since the eigenfunction expansion is nonlocal. However, we will show that it is possible to determine u locally from the functions f, g given on a set, which is bounded both in x and t. In case $Q \equiv 0$, our algorithm can be formulated in terms of analytic expressions.

The research was supported by the RFBR grant № 20-01-00627-a.

Features of the multipole scattering of acoustic field by refractive and absorbing inhomogeneities with small wave size

Dmitriev K.V., Rumyantseva O.D.
Moscow State University, Acoustics Department, Leninskie gory, 1, 2, Moscow, 119992
e-mail: kdmitrie@lanat.ru, burov@phys.msu.ru

The process of acoustic field scattering by a single inhomogeneity, which is placed in a homogeneous non-absorbing background medium, is considered. In this case, the scattered field can be represented as an expansion by components of different multipolarity. If the size of the inhomogeneity is small compared with the wavelength, then only the monopole and dipole components are significant in this expansion. To describe them, one can introduce complex scattering coefficients [1, 2]. These coefficients cannot be arbitrary, since they reflect the totality of all multiple scattering processes within the inhomogeneity under consideration. This is manifested as a relationship between the amplitude and phase of each scattering coefficient.

If the inhomogeneity is non-absorbing, the geometrical location of the points corresponding to the values of each of the scattering coefficients on the complex plane is a circumference with the center at the point -2i and with the radius 2. In the presence of absorption, this is the region inside the described circumference.

The inverse problem is posed of determining the parameters of the inhomogeneity of a small wave size on the base of the scattering data. To solve this problem, the Novikov algorithm [3, 4] is used. It turns out that for different values of the scattering coefficients, quality of the inhomogeneity reconstruction occurs different. It is well known that the reconstruction difficulty increases with the growth of the scatterer's "strength", which can be associated with the absolute value of the scattering coefficient. Inhomogeneities with small wave size and low contrast are usually reconstructed better than inhomogeneities with high contrast. However, this is not the only factor, and the phase of the scattering coefficient has a strong influence on the reconstruction result. The stability of the reconstruction of inhomogeneities with the positive real part of the scattering coefficients (the right side of the circumference on the complex plane) is greater than that of inhomogeneities with the negative real part (left side of the circumference). This can be explained by the defocusing of the field inside the inhomogeneity in the first case and by the focusing in the second case, which is associated, respectively, with a greater or lesser speed of sound inside the inhomogeneity compared to the speed of sound in the background medium.

The study was carried out with the grant from the Russian Science Foundation (project № 19-12-00098).

References

[1] V. A. Burov, S. A. Morozov, Relationship between the amplitude and phase of a signal scattered by a point-like acoustic inhomogeneity, *Acoustical Physics*, **47**(6), 659–664 (2001).

[2] K. V. Dmitriev, Matrix Green's functions and their application in analyzing scattering by density and sound velocity inhomogeneities, *Acoustical Physics*, **61**(6), 623–635 (2015).

[3] R. G. Novikov, Rapidly converging approximation in inverse quantum scattering in dimension 2, *Physics Letters A*, **238**(2-3), 73–78 (1998).

[4] R. G. Novikov, Approximate inverse quantum scattering at fixed energy in dimension 2, *Proc. V. A. Steklov Inst. Math.*, **225**(2), 285–302 (1999).

Lagrangian manifolds and a "naive" constructive approach for computing asymptotics via special functions near caustics

Dobrokhotov, S.Yu.
Ishlinsky Institute for Problems in Mechanics RAS and Moscow Institute of Physics and Technology, 101-1 prosp.Vernadskogo, Moscow, 119526, Russia
e-mail: s.dobrokhotov@gmail.com

We discuss the problem about the construction of effective asymptotic formulas near caustics. Very often, exact and asymptotic solutions to many problems for linear differential and pseudodifferential equations of mathematical physics are expressed in terms of elementary and special functions in parametric form. We show that the convenient parameters here are coordinates on suitable Lagrangian manifolds, and they also work effectively in the vicinity of caustics (Lagrangian singularities). Their use and simple "naive" approach make it possible to construct effective uniform asymptotics in the form of special functions of a complex argument. We illustrate our approach by examples from the linear water wave theory.

The talk is based on the results of joint work with V. Nazaikinski, S. Shlosman, A. Anikin, A. Tolchennikov, D. Minenkov, A. Tsvetkova, and supported by projects RSF (№ 16-11-10282) and RFBR (№ 17-51-150006, 17-01-00644).

Field structures in a laser with saturable absorption and doughnut aperture

Fedorov S.V., **Rosanov N.N.**, Veretenov N.A.
Ioffe Institute, Politekhnicheskaya str. 26, St. Petersburg 194021, Russia
e-mail: sfedorov2006@bk.ru, nnrosanov@mail.ru, torrek@gmail.com

Recently, there has been growing interest in topological structures in various branches of not only mathematics, but also physics and other natural sciences. Pioneering research of L. D. Faddeev [1] of topological three-dimensional structures of classical fields belonged to conservative systems (in the absence of energy dissipation). Dissipative systems in which a balance of energy input and output is realized are more stable. Various examples of such solitons in a laser medium with saturable absorption are presented in [2] and the literature cited there.

Here we present an analysis of another laser scheme, in which even one-dimensional topological solitons are also possible, but in a "topological" laser scheme. Namely, a fast saturable absorption laser cavity has a doughnut-shaped aperture with a wide hole. Radiation propagates predominantly normally to highly reflective mirrors, so that the ring is bypassed for a large number of consecutive reflections. In the approximations of quasioptics and the mean field, the dimensionless governing equation for a slowly varying field envelope E and the periodic boundary condition have the form

$$\frac{\partial E}{\partial t} = (i+d)\frac{\partial^2 E}{\partial x^2} + f(|E|^2)E, \quad E(x+L,t) = E(x,t).$$

Here t is time, the evolution variable, x is the coordinate along the laser aperture perimeter equal to L, d is the coefficient of effective diffusion, $0 < d \ll 1$. The function f of intensity $I = |E|^2$ in the simplest case is real:

$$f(I) = -1 - \frac{a_0}{1+I} + \frac{g_0}{1+I/\beta}$$

with positive coefficients of resonance absorption a_0, amplification g_0, and saturation β. Under the conditions of classical bistability, the function $f(I)$ vanishes for two positive values of the argument.

We assume that the trivial solution $E = 0$ is stable with respect to small perturbations, which is achieved for $f_0 < 0$. In addition to it, conditions for the existence and stability of regimes with a uniform intensity distribution and an inclined wavefront of the form $E = A_h \exp(iKx - i\Omega t)$, $A_h = const \neq 0$ are obtained. Here the inclination K_m is determined by the relation $K_m = 2\pi m/L$ with the integer topological index m. The set of numbers m is limited by the condition $dK_m^2 < \max f(I)$.

Of greatest interest are steady-state structures with an inhomogeneous intensity distribution, which, when $L \to \infty$, transform into laser solitons. For them $E(x,t) = F(\xi)\exp(-i\nu t)$, $\xi = x - Vt$. Here the frequency shift ν and velocity V are eigenvalues of the nonlinear problem

$$(i+d)\frac{d^2F}{d\xi^2} + V\frac{dF}{d\xi} + \big[i\nu + f(|F|^2)\big]F = 0, \quad F(0) = F(L), \quad dF/d\xi(0) = dF/d\xi(L).$$

The periodic function F is characterized by the phase incursion $\Delta\Phi = \int_0^L (d\arg E/dx)\, dx = 2\pi m$ with the topological index m. The structures with $m = 0$ are symmetric and motionless, whereas for $m \neq 0$ they are asymmetric and moving. We present a number of such stable structures for different topological indices.

The research was supported by the Russian Foundation for Basic Research, grant № 19-12-50174, and the Russian Scientific Foundation, grant № 18-12-00075.

References

[1] L. D. Faddeev, *Quantization of Solitons*, Preprint No. IAS-75-QS70, Institute for Advanced Study, Princeton, N.J., 1975.

[2] N. N. Rosanov, S. V. Fedorov, N. A. Veretenov, Topological transformations of three-dimensional dissipative solitons in the framework of the generalized Ginzburg–Landau equation, *Theoretical and Mathematical Physics*, **203**, 547–560 (2020).

On the density of states of the almost Mathieu operator in semiclassical approximation

A. Fedotov
St. Petersburg State University, 3, Ulyanovskaya st., St. Petersburg, 198504, Russia
e-mail: a.fedotov@spbu.ru

E. Shchetka
St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences, 27, Fontanka, St. Petersburg, Russia
e-mail: shchetka.ekaterina@mail.ru

First, we remind the definition and properties of the integrated density of states for ergodic operators (e.g., operators with random or almost periodic coefficients). Then, we briefly remind some basic properties of the almost Mathieu operator, and finally we describe the graph of its integrated density of states in the case when the operator can be studied in the semiclassical approximation, and discuss its Cantorian structure.

Medical ultrasound tomography problem: simulation with attenuation

Filatova V.M.
IKBFU, Kaliningrad, Russia
e-mail: vifilatova@kantiana.ru

The paper is devoted to the numerical study of the acoustical medium visualization in a formulation close to medical ultrasound tomography. Numerical experiments of visualization of the speed of sound and attenuation using Energy Reverse time migration (Energy RTM) were realized. The final image contains both the image of the speed of sound and the image of the attenuation. Analyzing the data of the inverse problem (pressure measurements on the boundary for different positions of the sources) these two images are separated.

Developing an algorithm for solving the inverse problem was supported by the Russian Science Foundation (grant № 16-11-10027). Numerical experiment using the OMICS compute cluster was supported by the Volkswagen Foundation. The authors acknowledge computational support from the OMICS compute cluster at the University of Lübeck.

Visualization of complex shape inclusions in the breast ultrasound tomography problem

Filatova V.M.
IKBFU, Kaliningrad, Russia
e-mail: vifilatova@kantiana.ru

Sedaikina V.A.
Saint Petersburg, Russia
e-mail: vsedaikina@gmail.com

Visualization of the acoustical medium is an important diagnostic instrument of determining breast cancer. The acoustical medium is visualized using the well-known geophysical method Reverse time migration (RTM). We describe some results of simulations of this problem for a specific breast model in 2D. The numerical solution of the problem relies on standard library of parallel computing (OpenMP) and is carried out on a cluster system. After applying the RTM procedure, the images are processed for localizing regions of interests (ROIs). The procedure of processing includes increasing

the intensity of the inclusions that interest us, filtering to reduce and remove noise, edge detection of the inclusions. The described method is automated to eliminate the need for manual processing.

Developing an algorithm for solving the inverse problem was supported by the Russian Science Foundation (grant № 16-11-10027). Numerical experiment using the OMICS compute cluster was supported by the Volkswagen Foundation. The authors acknowledge computational support from the OMICS compute cluster at the University of Lübeck.

Double-deck structure of boundary layers in weakly compressible heat-conducting flows in channels with heated wavy walls

Gaydukov R.K., **Fonareva A.V.**
National Research University Higher School of Economics, Moscow, Russia
e-mail: `roma1990@gmail.com, fon-alisa@yandex.ru`

The flow of an incompressible or a compressible fluid (gas) along surfaces with small irregularities has been widely considered in the literature. The well-known solutions of these problems for large Reynolds numbers are the flow with multi-deck (double- and triple-deck) structures of boundary layers. These solutions were found in various problems of liquid flows along surfaces with small irregularities, see, e.g., [1–3]. But all these problems were considered without taking into account the heat-transfer processes.

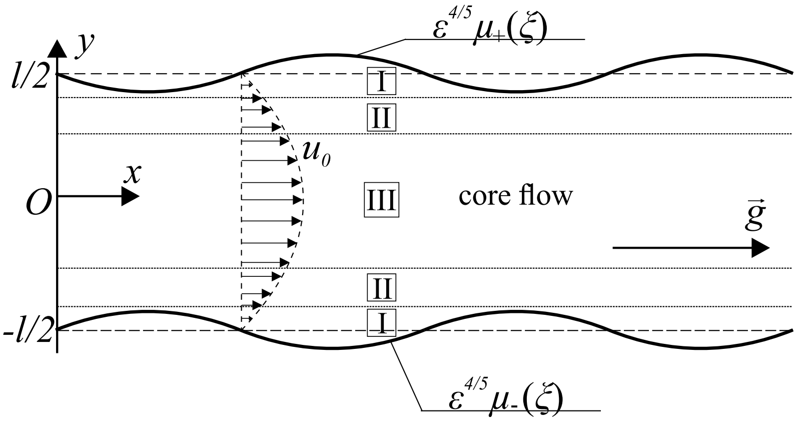


Fig. 1: 2D-channel with wavy walls.

We consider a nonstationary problem of a viscous weakly compressible heat-conducting fluid flow in a two-dimensional channel with small periodic irregularities on the heated walls for large Reynolds numbers Re, see Fig. 1. This problem is described by the system of Navier–Stokes, heat transfer and continuity equations in the Boussinesq approximation which has the form (in the dimensionless case)

$$\begin{aligned}
&\varepsilon^{2/5}\frac{\partial \mathbf{U}}{\partial t} + \langle \mathbf{U}, \nabla\rangle \mathbf{U} = -\nabla p + \varepsilon^2 \Delta \mathbf{U} + \mathbf{j}\beta\varepsilon^{2/5} T,\\
&\varepsilon^{2/5}\frac{\partial T}{\partial t} + \langle \mathbf{U}, \nabla T\rangle = \frac{\varepsilon^2}{Pr}\Delta T,\\
&\langle \nabla, \mathbf{U}\rangle = 0,
\end{aligned}$$

where $\mathbf{U} = (u, v)$ is the velocity vector, p is the pressure, T is the relative temperature, $\varepsilon = Re^{-1/2}$ is a small parameter, $\mathbf{j} = (1, 0)$, and β is the dimensionless coefficient of thermal expansion. We assume that the Prandtl number Pr is of order 1. This system is supplemented with nonslip boundary conditions on the walls and a given temperature on the walls. The coefficient at the time-derivative is determined by a selected time scale, see [2, 3].

We assume that the core flow inside is the Poiseuille flow (except for boundary layers). A formal asymptotic solution with double-deck structure of boundary layer was constructed using the boundary layer expansion method and the averaging method. To illustrate the results obtained, a numerical simulation of the flow in the thin boundary layer was carried out.

The study was implemented in the framework of the Basic Research Program at the National Research University Higher School of Economics (HSE University) in 2020 and was supported by Russian Science Foundation grant № 19-71-10003.

References

[1] A. V. Fonareva, R. K. Gaydukov, A compressible fluid flow with double-deck structure inside an axially symmetric wavy-wall pipe, *Russian Journal of Mathematical Physics*, **26**(3), 334–343 (2019).

[2] R. Gaydukov, V. Danilov, Multideck structures of boundary layers in compressible flows, *Proc. Int. Conf. DAYS on DIFFRACTION 2019*, 51–56 (2019).

[3] R. Gaydukov, V. Danilov, Equations for velocity oscillations in problems of a fluid flow along a plate with small periodic irregularities on the surface for large Reynolds numbers, *Proc. Int. Conf. DAYS on DIFFRACTION 2018*, 118–123 (2018).

Air-coupled ultrasonic inspection of anisotropic composite plates

Glushkov E.V., Glushkova N.V., **Miakisheva O.A.**
Institute for Mathematics, Mechanics and Informatics, Kuban State University, Krasnodar
e-mail: `evg@math.kubsu.ru, miakisheva.olga@gmail.com`

Carbon fiber-reinforced polymer composite materials (CPRF) are widely used in various engineering applications due to their advantage over other structural materials. Various dynamic and static loads and harsh environmental conditions cause degradation of their mechanical properties resulting in increased service costs and possible structural failure. One of the methods of non-destructive ultrasonic inspection of such structures is the use of contactless air-coupled transducers as a cheaper alternative to the laser Doppler vibrometry. To optimize their design and modes of use, computer simulation of wave generation in anisotropic laminate plates immersed in acoustic medium (air or fluid) is required.

The present work is focused on the study of the source-generated guided waves (GWs) propagating along the plate. Their characteristics depend on the material properties of the samples and so can be used as a convenient tool for material characterization and structural health monitoring. To simulate the excited ultrasonic GWs, an analytically based computer model for the source beam interaction with a fluid-loaded anisotropic laminate plate has been developed. The Green's matrix for the 3D source-fluid-plate system has been derived in terms of inverse Fourier transform integrals for elastic structures of various complexities, starting from the classical isotropic elastic layer up to generally anisotropic laminate composite materials.

The numerical examples show how the classical traveling Lamb waves transform into leaky GWs due to fluid loading, while the appearing undamped Scholte–Gogoladze waves carry some part of source energy along the plate to infinity instead of the Lamb waves. The manifestation of the backward mode and zero group velocity (ZGV) effects under fluid loading as well as the transformation of GW angular diagrams due to the plate's anisotropy are discussed. For the bulk acoustic waves reflected from and transmitted through the composite plate, the angular and frequency dependences of the reflection and transmission coefficients are also analyzed.

GPU-based optimizations of the boundary integral-equation method to solve direct and inverse diffraction problems

Leonid I. Goray
Alferov University, Khlopin str. 8/3, let. 'A', St. Petersburg, 194021, Russian Federation;
Institute for Analytical Instrumentation, Rizhsky pr. 26, St. Petersburg, 190103, Russian Federation
e-mail: `lig@pcgrate.com`

Alexander S. Dashkov
Alferov University, Khlopin str. 8/3, let. 'A', St. Petersburg, 194021, Russian Federation
e-mail: `dashkov.alexander.om@gmail.com`

The diffraction grating problem with one-periodical gratings (2D structures) with arbitrary conductivity is considered. The oblique-incident time-harmonic radiation is assumed. Several significant optimization techniques were introduced in the literature, in particular, short-wave problems can be effectively solved using the Boundary Integral Equation Method ([1], Ch. 12). Nonetheless, the methods are not superb in terms of speed for randomly-rough surfaces and inverse problem solutions. In a particular case, a graphical processing unit (GPU) utilization suggested for performance improvement. The optimization is done via the transfer of most time-consuming numerical operations to GPU including calculus of Green's functions and their derivatives. The GPU-based code is written with CUDA [2] and Open CL [3] technologies.

The accelerated method was tested on direct and inverse diffraction problems in the X-ray wavelength range. The rough surface statistics is required to rigorously compute the scattering intensity using PCGrate™ software [4]. We have generated rough boundaries with the Gaussian distribution of heights and Gaussian-like correlation function. To calculate specular reflectances using the direct electromagnetic solver, we employ the scattering matrices approach and the Monte Carlo simulations to average the statistics owing to individual surfaces over an ensemble of realizations. The inverse diffraction problem was solved using a genetic algorithm. The accelerated method has shown improvements up to several times.

References

[1] E. Popov (Ed.), *Gratings: Theory and Numeric Applications*, 2nd rev. ed., Presses Universitaires de Provence, Marseille (2014).
[2] What is CUDA? `https://www.nvidia.com/object/io_69526.html`, Accessed: 2020-03-07.
[3] OpenCL Overview, `https://www.khronos.org/opencl/`, Accessed: 2020-03-07.
[4] I.I.G., Inc. Website, `https://www.pcgrate.com/index_html`, Accessed: 2020-03-07.

Exponential dichotomy of linear cocycles over irrational rotations

Ivanov A.V.
St. Petersburg State University, Universitetskaya nab. 7/9, St. Petersburg, 199034, Russian Federation
e-mail: `a.v.ivanov@spbu.ru`

We study a skew-product map

$$F_A : \mathbb{T}^1 \times \mathbb{R}^2 \to \mathbb{T}^1 \times \mathbb{R}^2 \tag{1}$$

defined for any $(x, v) \in \mathbb{T}^1 \times \mathbb{R}^2$ by

$$(x, v) \mapsto (\sigma_\omega(x), A(x)v),$$

where $\sigma_\omega(x) = x + \omega$ is a rotation of a circle $\mathbb{T}^1$ with irrational frequency ω and

$$A : \mathbb{T}^1 \to SL(2, \mathbb{R})$$

is a measurable function with respect to the Haar measure. It is supposed that the transformation A has a special form. Particularly, given a C^2-function $f : \mathbb{T}^2 \to \mathbb{R}$ of the two-torus, define sets

$$\mathcal{D}_+ = \{z \in \mathbb{T}^2 : f(z) > 0\}, \quad \mathcal{D}_- = \{z \in \mathbb{T}^2 : f(z) < 0\}$$

and for any $x \in \mathbb{T}^1$ consider its intersections $S_\pm(x) = \mathcal{D}_\pm \cap I(x)$ with a segment $I(x) = \{(s, x+\omega s), s \in \mathbb{T}^1\} \subset \mathbb{T}^2$. Represent $S_\pm(x)$ as

$$S_\pm(x) = \bigcup_{k=1}^{N(x)} \Delta_k^\pm,$$

where $\Delta_k^\pm$ are connected components ordered in a natural way with respect to increase of the parameter s. Then the transformation A is assumed to be of the form

$$A(x) = R(\varphi_{N(x)}) \circ Z(\alpha_{N(x)}) \circ R(\varphi_{N(x)-1}) \circ Z(\alpha_{N(x)-1}) \circ \ldots \circ R(\varphi_1) \circ Z(\alpha_1),$$

where

$$R(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}, \quad Z(\alpha) = \begin{pmatrix} \mathrm{e}^{\alpha} & 0 \\ 0 & \mathrm{e}^{-\alpha} \end{pmatrix}$$

and

$$\varphi_k = \mu \int_{\Delta_k^-} |f|^{1/2} \mathrm{d}l, \quad \alpha_k = \mu \int_{\Delta_k^+} |f|^{1/2} \mathrm{d}l, \quad \mu \gg 1.$$

Such linear cocycle appears as a model in the problem of reducibility for the Schrödinger equation with quasiperiodic potential f in the adiabatic limit [1].

In the present work, using the ideas similar to [2], we show that there exists sufficiently large $\mu_0 > 0$ and a subset $\mathcal{E}_h \subset (\mu_0, \infty)$ such that

1. for any $\mu_1 < \mu_0$ the Lebesgue measure $\mathrm{leb}\,((\mu_1, \infty) \setminus \mathcal{E}_h) = O\left(\mathrm{e}^{-c\mu_1}\right)$ with some positive constant c;
2. for any $\mu \in \mathcal{E}_h$ the system (1) possesses the exponential dichotomy.

References

[1] A. V. Ivanov, Connecting orbits near the adiabatic limit of Lagrangian systems with turning points, *Reg. & Chaotic Dyn.*, **22**(5), 479–501 (2017).

[2] V. F. Lazutkin, Making fractals fat, *Reg. & Chaotic Dyn.*, **4**(1), 51–69 (1999).

Two-component optical vortices at zero group velocity dispersion

Kalinovich A.A., Sazonov S.V., Bryantsev D.S., Komissarova M.V., Zakharova I.G.
Faculty of Physics, Moscow State University, Moscow, 119991 Russia
e-mail: `kalinovich@gmail.com, sazonov.sergey@gmail.com, brisbryantsev@mail.ru, komissarova@physics.msu.ru, zaharova@physics.msu.ru`

In recent decades, soliton structures have attracted the attention of researchers. Among them, solitons with phase singularities are especially important. They are also called vortex solitons. Such solitons are characterized by azimuthal instability, which usually leads to their decay into many irrotational sub-solitons. A number of works in the field of nonlinear optics devoted to the study of the formation and propagation of vortex solitons in various media have revealed some regularities [1, 2]. It was found that their stability is achieved in the presence of several competing processes, some of which trying to localize the pulse and others trying to expand it. Media with required characteristics may be those with third and fifth order nonlinearities. Besides that media with nonlinear modulation, liquid crystals, and waveguide arrays can possess the required characteristics.

In this paper, using numerical simulation, we study the formation and propagation of vortex solitons in a medium with competing group velocity dispersion (GVD) and third-order dispersion. A system of equations, describing the considered process in the case of group and phase synchronism, looks as follows:

$$i\frac{\partial\psi_1}{\delta z} = -\frac{\beta_2^{(1)}}{2}\frac{\partial^2\psi_1}{\partial\tau^2} + \alpha_1\psi_1^*\psi_2 + \frac{c}{2n_1\omega}\Delta_\perp\psi_1, \tag{1}$$

$$i\frac{\partial\psi_2}{\delta z} = -\frac{\beta_2^{(2)}}{2}\frac{\partial^2\psi_2}{\partial\tau^2} - i\frac{\beta_3^{(2)}}{6}\frac{\partial^3\psi_2}{\partial\tau^3} + \alpha_2\psi_1^2 + \frac{c}{4n_2\omega}\Delta_\perp\psi_2, \tag{2}$$

where $\beta_2^{(1)}$, $\beta_2^{(2)}$ are the coefficients associated with GVD and third-order dispersion in a nonlinear medium. Firstly we consider anamalous GVD $\beta_2^{(1)} < 0$. At that, $\beta_3^{(2)} > 0$. The initial profile is given in vortex form

$$\psi_{1,2}(x, y, \tau, z = 0) = \psi_{10,20}(x + imy)e^{-x^2-y^2-\tau^2}, \tag{3}$$

where m is topological charge of vortex.

During the study of the existence of stable vortex-soliton solutions, we consider cases when $\beta_2^{(2)} = 0$; $\beta_2^{(2)} > 0$, $|\beta_2^{(2)}| \ll |\beta_2^{(1)}|$ and $\beta_2^{(2)} < 0$, $|\beta_2^{(2)}| \ll |\beta_2^{(1)}|$. If it is possible to find stable solutions in the above mentioned cases, it is planned to complicate the model, namely, to introduce phase and group mismatch. Then it is interesting to go beyond the framework of anomalous GVD, considering transversal inhomogeneity of the medium.

The investigation was made using support of the Russian Science Foundation (Grant № 17-11-01157).

References

[1] D. Mihalache, Stable spinning optical solitons in two and three dimensions, *Opto-Electronics Review*, **13**(2), 123–127 (2005).

[2] H. Michinel, J. Campo-Taboas, M. L. Quiroga-Teixeiro, J. R. Salgueiro, R. Garcıa-Fernandez, Excitation of stable vortex solitons in nonlinear cubic-quintic materials, *Journal of Optics B: Quantum and Semiclassical Optics*, **3**, 314–317 (2001).

Asymptotics of saddle-node bifurcation

Kalyakin L.A.
Institute of Mathematics with Computing Center, Ufa Federal Research Center, Russian Academy of Sciences, Ufa, 450008 Russia
e-mail: `klenru@mail.ru`

We consider a model differential equation of the second order, with slowly varying parameter,

$$\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} = (x^2 + \varepsilon t)(1 - x), \quad t > 0,\ 0 < \varepsilon \ll 1. \tag{1}$$

Under frozen parameter, the corresponding autonomous equation has equilibriums: saddle and stable nodes. When parameter is deforming, pair saddle-node is joining. Asymptotic solution is constructed near such dynamic bifurcation. It is discovered that in a narrow transient layer the main term of asymptotics is determined by Riccati and KPP equations. The main result is determination of the shift of the transient layers from moment of the bifurcations. Exact statement are illustrated by numerical experiments [1].

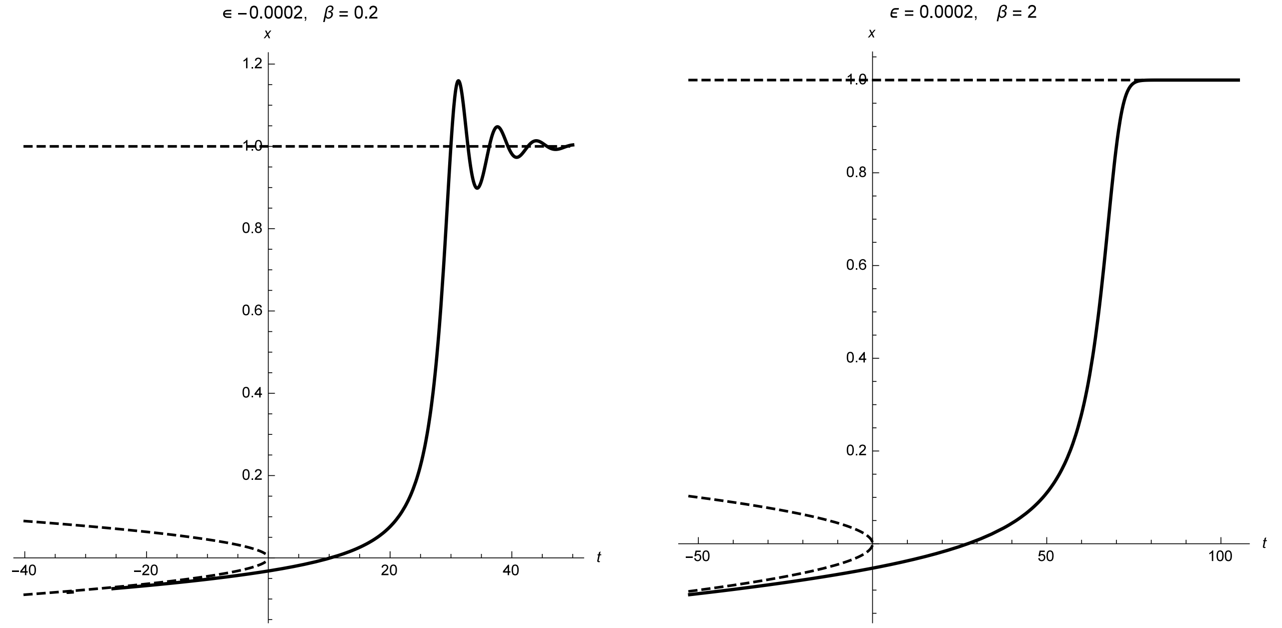


Fig. 1: Numerical experiments with Eq.(1) for different β. The graphs of the solution asymptotically coincide with the parabola $-\sqrt{-\varepsilon t}$ as $t \to \infty$.

References

[1] L. A. Kalyakin, Asymptotics of the solution of a differential equation in a saddle-node bifurcation, *Computational Mathematics and Mathematical Physics*, **59**(9), 1454–1469 (2019).

On the adiabatic sound propagation in a shallow sea with a curved underwater canyon

Kazak M.S., Petrov P.S.
V. I. Il'ichev Pacific Oceanological Institute, 43 Baltiyskaya str., Vladivostok, 690041, Russia
e-mail: kazak.ms@poi.dvo.ru, petrov@poi.dvo.ru

Sound propagation is a cylindrically-symmetric shallow-water waveguide is considered. The bottom relief in cylindrical coordinates (r, θ, z) is described by the function

$$h(r) = \begin{cases} h_0 + \Delta H \sin\Big(\pi \dfrac{r - r_1}{r_2 - r_1}\Big), & \text{if } r_1 < r < r_2, \\ h_0, & \text{otherwise,} \end{cases} \tag{1}$$

where r_1 and r_2 are canyon boundaries, ΔH is the canyon depth and h_0 is the depth outside the canyon. The waveguide consists of the water column $0 \le z \le h(r)$ and the penetrable bottom $z > h(r)$. Sound propagation is described by the Helmholtz equation

$$\frac{1}{r}\left(rP_r\right)_r + \frac{1}{r^2}P_{\theta\theta} + P_{zz} + \frac{\omega^2}{c^2}P = -\frac{1}{r}\delta(r - r_s)\delta(\theta - \theta_s)\delta(z - z_s). \tag{2}$$

The pressure-release condition for acoustic pressure $P(r, \theta, z)$ is imposed at the surface $z = 0$

$$P|_{z=0} = 0.$$

According to the normal mode theory, the sound pressure in the water column can be written in the form of truncated modal expansion $P(r, \theta, z) \approx \sum_{j=1}^{N} A_j(r, \theta)\phi_j(z, r, \theta)$, where ϕ_j are mode functions obtained from the solution of the acoustical spectral problem [1] and A_j are modal amplitudes. The latter are obtained (in adiabatic approximation) by solving horizontal refraction equations [1]

$$\frac{\partial^2 A_m(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial A_m(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A_m(r,\theta)}{\partial \theta^2} + A_m(r,\theta)k_m^2 = -\frac{1}{r}\frac{\phi_m(r,\theta,z_s)}{\rho}\delta(r-r_s)\delta(\theta-\theta_s)\,. \quad (3)$$

The method of sound pressure field calculation in a waveguide with rotational symmetry is described in this work. The separation of variables method is applied to the solution of the horizontal refraction equations. The solution can be represented in the form

$$A_m(r,\theta) = \sum_{j=0}^{N_\theta} Q_{mj}(r)\psi_j(\theta)\,, \quad (4)$$

where functions $\psi_j(\theta)$ are obtained from the solution of the the Sturm–Liouville problem in the angular variable (with the periodicity conditions), and $Q_{m,j}(r)$ are radial modes that can be represented outside the ring $[r_1, r_2]$ as

$$Q_{mj}(r) = \alpha_{mj} J_j(k_m r),\ \text{if } r \le r_1\,, Q_{mj}(r) = \beta_{mj} H_j^{(1)}(k_m r),\ \text{if } r \ge r_2\,, \quad (5)$$

where $J_j(r)$ and $H_j^{(1)}(r)$ are the Bessel and the Hankel functions respectively. The values of the function $Q_{mj}(r)$ inside the ring can be computed numerically. After that the sound pressure field can be computed by formulae (3) and (4) (an example is shown in Fig. 1).

The solution properties of horizontal refraction equations in the case of underwater canyon are described using the ray theory. The relation between the location of the source and the amount of acoustical energy trapped above the canyon is studied. A correspondence between the indices of Bessel and Hankel functions in (5) and the horizontal rays trapped in the canyon area is established.

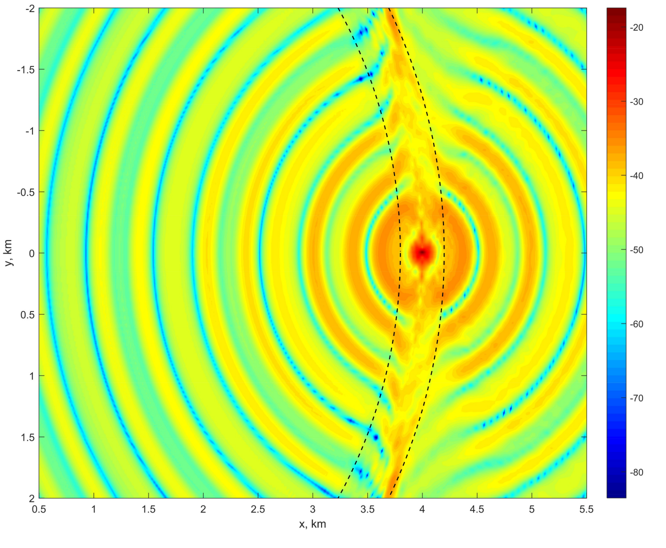


Fig. 1: Contour plot of the sound pressure field (in dB re 1 m). The inner and outer boundaries of the canyon are shown by dashed lines. Note the localization of acoustical energy over the canyon.

The reported study was supported by the Russian Foundation for Basic Research under the contracts № 18-05-00057_a and № 18-35-20081_mol_a_ved.

References

[1] F. B. Jensen, W. A. Kuperman, M. B. Porter, H. Schmidt, *Computational Ocean Acoustics*, Springer Science & Business Media, New York, 2011.

Inverse problem for a linearized model of oxygen transport in brain

Kovtanyuk A.E.[1,2], Chebotarev A.Yu.[1], Turova V.L.[2], Sidorenko I.N.[2], Lampe R.[2]
[1]Institute for Applied Mathematics, FEB RAS, Vladivostok, Russia
[2]Technische Universität München, München, Germany
e-mail: `kovtanyuk.ae@dvfu.ru`

A two-compartment (blood and tissue) model of oxygen transport is considered. It is assumed that the both compartments occupy the same spatial region $\Omega \subset \mathbb{R}^3$ and have different volume fractions for the blood and tissue compartments, σ and $1-\sigma$, respectively. Following [1, 2], the oxygen transport can be described by the following coupled equations:

$$-\alpha\Delta\varphi + \mathbf{v}\cdot\nabla\varphi = G + \sum_{j=1}^{m} q_j f_j, \quad -\beta\Delta\theta = -\kappa G - \mu + \sum_{j=1}^{m} p_j f_j, \quad x \in \Omega. \tag{1}$$

Here, φ and θ are the blood and tissue oxygen concentrations, respectively; μ describes the tissue oxygen consumption; $G = c(\theta - \psi)$ is the intensity of oxygen exchange between the blood and tissue fractions, where ψ is the plasma oxygen concentration; $\kappa = \sigma(1-\sigma)^{-1}$, where σ is the volumetric fraction of vessels; $\mathbf{v}$ is a prescribed continuous velocity field in the entire domain G; α and β are diffusivity parameters of the corresponding phases; f_j are the characteristic functions of the disjoint subdomains $\Omega_j \subset \Omega$, $j = 1, ..., m$, which are some neighborhoods of the ends of arterioles and venules. That is the contribution from arterioles and venules are described by the source functions of equations (1) with unknown intensities q_j, p_j, $j = 1, ..., m$. Notice that, in [1], this contribution is described by the appropriate boundary conditions.

There are nonlinear monotonic dependencies of the tissue oxygen metabolic rate, μ, on the tissue oxygen concentration, θ, and of the plasma oxygen concentration, ψ, on the blood oxygen concentration, φ. To simplify the model, the following linear approximations: $\mu = p\theta + s$ and $\psi = a\varphi + b$, where $a, p > 0$, are used.

Equations (1) are supplemented by the following boundary conditions imposed on $\Gamma = \partial\Omega$:

$$\alpha\partial_n\varphi + \gamma(\varphi - \varphi_b)|_\Gamma = 0, \quad \beta\partial_n\theta + \delta(\theta - \psi_b)|_\Gamma = 0. \tag{2}$$

Here, ∂_n denotes the outward normal derivative at points of the domain boundary. Nonnegative functions φ_b, ψ_b, γ, and δ are given.

The *Inverse Problem* consists in finding intensities $r = (q_1, ..., q_m, p_1, ..., p_m) \subset \mathbb{R}^{2m}$ and the corresponding solution $y = (\varphi, \theta)$ of the boundary-value problem (1), (2) with the following integral overdetermination:

$$\int_{\Omega_j} \varphi dx = Q_j, \quad \int_{\Omega_j} \theta dx = P_j, \quad j = 1, ..., m.$$

Here, Q_j and P_j are the prescribed avereged values of the functions φ, θ with respect to subdomains Ω_j.

The unique solvability of the inverse problem is proven, an algorithm to find solutions is proposed, and conducted numerical experiments are discussed.

This work is supported by the Russian Foundation of Basic Research (project № 20-01-00113_a) and by the Klaus Tschira Foundation.

References

[1] A. E. Kovtanyuk, A. Yu. Chebotarev, A. A. Dekalchuk, N. D. Botkin, R. Lampe, Analysis of a mathematical model of oxygen transport in brain, *Proc. Int. Conf. Days on Diffraction 2018*, 187–191 (2018).

[2] A. E. Kovtanyuk, A. Yu. Chebotarev, N. D. Botkin, V. L. Turova, I. N. Sidorenko, R. Lampe, Continuum model of oxygen transport in brain, *J. Math. Anal. Appl.*, **474**, 1352–1363 (2019).

On the mode parabolic equation method for the elastic media

Kozitskiy S.B.
Il'ichev Pacific Oceanological Institute, 43 Baltiiskaya St., Vladivostok, 690041, Russia
e-mail: `skozi@poi.dvo.ru`

Mode parabolic equations appeared as a convenient tool for solving 3D problems of acoustics. Various parabolic equations for elastic media have been derived in some articles. As a rule, they have different restrictions. Previously, we have derived a system of mode parabolic equations taking into account the weak elasticity of the bottom [1].

In this work we develop approach to derivation of a system of mode parabolic equations (MPEs) for the 3D layered elastic media without limitation of the weak elasticity, based on the generalized multiscale expansions method. As a small parameter ϵ a ratio of a typical wavelength to a typical size of horizontal inhomogeneities has been chosen. We introduce slow variables $X = \epsilon x$ and $Y = \epsilon^{1/2} y$, and postulate decomposition for the elastic modules $E_{\mathrm{ef}} = \lambda + 2\mu = E_0(X, z) + \epsilon E_1(X, Y, z)$, $\mu = \mu_0(X, z) + \epsilon\mu_1(X, Y, z)$. Also for density we have $\rho^{-1} = \gamma_0(X, z) + \epsilon\gamma_1(X, Y, z)$. With the above expressions and parabolic anzats for the displacements $U = (u, v, w)^T$, we expand the equations of elastodynamics [2] and collect terms at the same powers of ϵ.

At $O(\epsilon^0)$ we obtain elastic self-adjoint eigenvalue problem $\mathbf{L}(k)\Phi = 0$ which is equivalent to the one in the work [3] ($\Phi = (\varphi, \psi)^T$ and k is eigenvalue):

$$(\gamma_0\phi_z)_z + \gamma_0\left(\omega^2 c^{-2} - k^2\right)\phi - \mathrm{i}k\gamma_0\nu_2\psi + \mathrm{i}k\delta\nu_3\alpha_0\gamma_0[(\mathrm{i}k\phi + \psi_z) - \mathrm{i}k\delta(\phi_z - \mathrm{i}k\psi)] = 0\,,$$

$$(\gamma_0\psi_z)_z + \gamma_0\left(\omega^2 c_s^{-2} - k^2\right)\psi + \mathrm{i}k\gamma_0\nu_1\phi - \mathrm{i}k\delta\nu_3\alpha_0\gamma_0[(\phi_z - \mathrm{i}k\psi) + \mathrm{i}k\delta(\mathrm{i}k\phi + \psi_z)] = 0\,.$$

Variables $P_0 = -E_0(w_{0z} + \mathrm{i}ku_0) = A(X, Y)\phi(X, z)$ and $S_0 = -\mu_0(u_{0z} - \mathrm{i}kw_0) = A(X, Y)\psi(X, z)$ describes compressional waves and shear waves of vertical polarization. Also $\alpha_1 = [1 - k^2\delta^2]^{-1}$, $c^2 = \gamma_0 E_0$, $c_s^2 = \gamma_0\mu_0$, $\nu_1 = \gamma_{0z}\gamma_0^{-1} + 2\mu_{0z}E_0^{-1}$, $\nu_2 = \gamma_{0z}\gamma_0^{-1} + 2\mu_{0z}\mu_0^{-1}$, $\nu_3 = \gamma_{0z}\gamma_0^{-1} + \mu_{0zz}\mu_{0z}^{-1}$. Here we introduced function $\delta(X, z) = 2\gamma_0\mu_{0z}\omega^{-2}$ and often $\|k\delta\| \ll 1$. So we can use perturbation theory to obtain eigenvalues and eigenfunctions from the problem with $\mu_{0z} = 0$ and $\gamma_{0z} = 0$. Countable solutions of the eigenvalue problem (Φ_j, k_j) are orthogonal to each other, normalized and indexed by $j = 1, 2\ldots$. At $O(\epsilon^{1/2})$ we get eigenvalue problem for the elastic waves of horizontal polarization.

At $O(\epsilon)$ we obtain inhomogeneous eigenvalue problem for variables P_1, S_1, with right part depending on P_0, S_0. Compatibility conditions of this problem take a form of MPEs. In the case of one-mode representation for P_i, S_i as a result we derived system of adiabatic MPEs for $j = M, \ldots, N$

$$2\mathrm{i}k_j\alpha_{1j}A_{j,X} + \mathrm{i}k_{j,X}\alpha_{2j}A + \alpha_{3j}A_{j,YY} + \alpha_{4j}A_j = 0\,.$$

Here α_{ij} have cumbersome form, but when $\mu_{0z} = 0$ and $\gamma_{0z} = 0$ for $i = 1, 2, 3$ we have $\alpha_{ij} = 1$. Also we can derive elastic MPEs with interacting modes from multi-mode representation for P_i, S_i in the same way as in the work [1].

An advantage of the presented approach for the derivation of the elastic MPEs is in possibility to use small parameter δ to obtain solutions of the elastic eigenvalue problems by the perturbation theory and to estimate values of coefficients α_{ij}.

References

[1] S. B. Kozitskiy, Examples of test calculations by the acoustic mode parabolic equation with the mode interaction and the elastic bottom, *Proc. Int. Conf. "Days on Diffraction 2019"*, IEEE, 105–110 (2019).

[2] V. M. Babich, A. P. Kiselev, *Elastic Waves: High Frequency Theory*, CRC Press, Boca Raton, 2018.

[3] M. B. Porter, E. L. Reiss, A numerical method for bottom interacting ocean acoustic normal modes, *J. Acoust. Soc. Am.*, **77**, 1760–1767 (1985).

Resonance scattering of the dipole radiation by a cylindrical density duct in a magnetoplasma

Kudrin A.V., Ivoninsky A.V., Ostafiychuk O.M.

Department of Radiophysics, University of Nizhny Novgorod, 23 Gagarin Ave., Nizhny Novgorod 603950, Russia

e-mail: kud@rf.unn.ru, alexandr.ivoninskiy@gmail.com, olegost09@rambler.ru

Zaboronkova T.M.

Department of Nuclear Physics, R. E. Alekseev Technical University of Nizhny Novgorod, 24 Minin St., Nizhny Novgorod 603950, Russia

e-mail: t.zaboronkova@rambler.ru

The problem of the interaction of electromagnetic waves with magnetic-field-aligned density irregularities, known as density ducts, in a magnetoplasma has a long history and has been studied by many authors. The interest in this problem has been motivated by the observations of whistler wave guidance by density ducts in the near-Earth space [1] and formation of field-aligned plasma structures in heating ionospheric [2] and laboratory [3] experiments. The possibility of resonance interaction of electromagnetic waves with such plasma structures is of primary importance for their diagnostics and leads to the necessity of a detailed study of conditions under which the scattering of electromagnetic waves from plasma density irregularities can exhibit resonance behavior. Most papers on the subject deal with the resonance scattering of plane electromagnetic waves by density ducts (see, e.g., [4] and references therein). It is the purpose of the present work to study the resonance scattering from a cylindrical density duct irradiated by the field of a localized electromagnetic source that is placed in a background magnetoplasma outside the duct.

We consider an electric dipole source which is parallel to the duct having decreased plasma density with respect to the background medium. It is shown that such a source, depending on its frequency, can excite plasmon resonances as well as resonances at the plasma frequencies of the outer and inner regions of the duct. Conditions have been found under which the dipole source also excites forward and backward eigenmodes of the duct, which give rise to complex waves at certain frequencies. It turns out that this circumstance can result in appearance of the previously unrevealed resonances of the scattered field, which are related to formation of standing-wave structures in the duct at the corresponding frequencies. We analyze how manifestation of this phenomenon depends on the source parameters and the azimuthal indices of the observed resonances. Numerical results will be reported for conditions typical of decreased density ducts in the ionospheric plasma.

Acknowledgments. This work was supported by the Ministry of Science and Higher Education of the Russian Federation within the framework of the state assignment to University of Nizhny Novgorod.

References

[1] R. A. Helliwell, *Whistlers and Related Ionospheric Phenomena*, Dover, Mineola, 2006.

[2] A. V. Gurevich, Nonlinear effects in the ionosphere, *Phys. Usp.*, **50**, 1091–1122 (2007).

[3] A. V. Kostrov, A. V. Kudrin, L. E. Kurina, G. A. Luchinin, A. A. Shaykin, T. M. Zaboronkova, Whistlers in thermally generated ducts with enhanced plasma density: Excitation and propagation, *Phys. Scr.*, **62**, 51–65 (2000).

[4] A. V. Ivoninsky, A. V. Kudrin, Resonance scattering of an extraordinary wave by a cylindrical density depletion in a magnetoplasma, *Phys. Plasmas*, **25**, 102112 (2018).

Interferometric localization of moving noise source by using of high-frequency signals

Kuznetsov G.N., Kuzkin V.M.
General Physics Institute of RAS, 38 Vavilova str., Moscow, Russia
e-mail: kumiov@yandex.ru

Kutsov M.V., Pereselkov S.A., Prosovetskii D.Yu.
Voronezh State University, Universitetskaya pl. 1, Voronezh, Russia
e-mail: pereselkov@yandex.ru

The results of an experimental and analytical study of the method for estimating the coordinates and noise source radial velocity by using of single cylindrical small-size vector-scalar antenna are presented. Frequency-time processing was used, consistent with the spatial interference structure formed by moving source by signals: directed and reflected from a free surface. Dependences on the bearing time, radial velocity, distance and depth of the source are obtained. A qualitative and quantitative explanation of the experimental data is given.

To estimate the range and depth, a two-beam model of interference structure of sound field is used. In accordance with the principles of interferometric processing, sound energy is focused on the hologram, which accumulates over a sufficiently long period of time. In addition, the radial velocity is estimated simultaneously with distance detection and estimation. The use of a vertically oriented vector-scalar antenna, despite its small diameter (18 cm), allows us to perform real-time bearing. Thus, the developed method makes it possible to estimete all parameters of a moving source taking into account long-term accumulation, i.e. with a significantly higher signal-to-noise ratio.

The work was supported by RFBR (grants № 19-29-06075 and № 19-08-00941). The research of D. Yu. Prosovetskii was supported by grant of President Russian Federation (№ MK-933.2019.8).

Spectral properties of a class of functional difference equations with meromorphic potential and applications

Mikhail A. Lyalinov
Department of Mathematical Physics, Institute of Physics, St. Petersburg University, Universitetskaya nab. 7/9, 199034, St. Petersburg, Russia
e-mail: lyalinov@yandex.ru

Svetlana V. Polyanskaya
North-West Institute of Management (RANHiGS), Sredniy ave. 57/43 VO, 199178, St. Petersburg, Russia

We study functional difference (FD) equations of the second order with a potential belonging to a special class of meromorphic functions. Essential and discrete spectrum as well as the behavior of eigenunctions are considered. To this end, a FD equation is reduced to an integral equation that admit an efficient study. Some applications to a Schrödinger operator with a singular potential having support on a conical surface are also addressed.

Symmetric guided waves in an isotropic inhomogeneous waveguide

Martynova V.Yu., Moskaleva M.A., Raschetova D.V.
Department of Mathematics and Supercomputing, Penza State University, Penza, 440026, Russia
e-mail: 79273698109@ya.ru, m.a.moskaleva1@gmail.com, dariaraschetova@gmail.com

Symmetric electromagnetic waves in a shielded plane dielectric slab was introduced and studied recently in [1], where the permittivity is anisotropic and homogeneous. In this report we study propagation of symmetric electromagnetic wave in a shielded plane dielectric slab characterised with a scalar inhomogeneous permittivity. We use the operator pencils theory to study the wave propagation problem [2].

Let $\Sigma := \{(x, y, z) : 0 \leqslant x \leqslant h, (y, z) \in \mathbb{R}^2\}$ be a slab having perfectly conducted walls

$$\sigma_0 := \{(x, y, z) : x = 0, (y, z) \in \mathbb{R}^2\}, \quad \sigma_h := \{(x, y, z) : x = h, (y, z) \in \mathbb{R}^2\},$$

that located in the Cartesian coordinates $Oxyz$.

We study the propagation of a monochromatic electromagnetic wave $(\mathbf{E}, \mathbf{H})e^{-i\omega t}$, where ω is the circular frequency and $\mathbf{E}, \mathbf{H}$ are the complex amplitudes, along the layer Σ.

The vector-valued functions $\mathbf{E}, \mathbf{H}$ have the form

$$\mathbf{E} = (e_x, e_y, e_z)^\top \cdot e^{i(\gamma_y y+\gamma_z z)}, \quad \mathbf{H} = (h_x, h_y, h_z)^\top \cdot e^{i(\gamma_y y+\gamma_z z)}, \tag{1}$$

where

$$\begin{aligned} e_x &\equiv e_x(x), \quad e_y \equiv e_y(x), \quad e_z \equiv e_z(x), \\ h_x &\equiv h_x(x), \quad h_y \equiv h_y(x), \quad h_z \equiv h_z(x), \end{aligned}$$

and γ_y, γ_z are unknown real spectral parameters (propagation constants of the guided wave), $(\,\cdot\,)^\top$ is the transposition operation. The field (1) is called symmetric guided waves [1].

The layer Σ is filled with nonmagnetic isotropic homogeneous medium characterised by the permittivity

$$\varepsilon \equiv \varepsilon_0 \varepsilon(x).$$

We also assume that inside the layer $\mu = \mu_0 > 0$ is the permeability of free space.

Maxwell's equations in the harmonic mode have the form

$$\operatorname{rot} \mathbf{E} = i\omega\mu\mathbf{H}, \quad \operatorname{rot} \mathbf{H} = -i\omega\varepsilon\mathbf{E}. \tag{2}$$

The main problem is to find coupled eigenvalues $(\gamma_y, \gamma_z) = (\widehat{\gamma}_y, \widehat{\gamma}_z)$ for which there exists a nontrivial field $(\mathbf{E}, \mathbf{H})e^{-i\omega t}$, where the functions $\mathbf{E}$, $\mathbf{H}$ given by (1) satisfy Maxwell's equations (2) and the tangential components of the electric field vanish at the boundaries $x = 0$, $x = h$. In addition, the energy finiteness condition must be fulfilled:

$$\int_0^h (\omega\mu\varepsilon|\mathbf{E}|^2 + |\mathbf{H}|^2)dx < \infty.$$

References

[1] E. O. Biteleva, D. V. Raschetova, D. V. Valovik, Fully symmetric guided electromagnetic waves in a shielded plane dielectric slab, *Lobachevskii Journal of Mathematics*, **40**(10), 1640–1652 (2019).

[2] Yu. Smirnov, *Mathematical Methods for Electromagnetic Problems*, Penza State University, Penza, 2009.

Wave effects in stochastic reaction-diffusion model of quorum-sensing in bacterial populations

Maslovskaya A.G.[1], Kuttler Ch.[2]

[1]Amur State University, Ignatyevskoe Shosse 21, 675027 Blagoveshchensk, Russia

[2]Munich Technical University, Boltzmannstr. 3, 85747 Garching, Germany

e-mail: `maslovskayaag@mail.ru`

In recent decades, significant attentions have been given using methods and techniques of mathematical modelling and computer simulation to predict behavior of complex biological systems.

Specifically, bacterial populations demonstrate ability to a multicellular organization and regulation of their reactions depending on changes in environment conditions. One of the possible mechanisms of bacterial communication can be realized by the so-called "quorum-sensing", which can cause reducing efficiency of antibacterial medicines.

The bacterial communication characteristics have been previously described mathematically by a system of ODEs taking into account bacterial population growth and delay effect, including positive and negative feedback in biological system [1]. Moreover, description of heterogeneous space distribution of signaling molecules produced by bacteria enables us to express model in a form of system of reaction-diffusion PDEs [1]. However, experimental data suggest the appearance of time-dependent fluctuations of signal substances providing quorum sensing during the process of nucleation and growth of bacterial population [2]. Therefore, we can modify the basic model in view of stochastic dynamics of bacterial population during the observation time. In the present study, we develop the stochastic reaction-diffusion modification of the bacterial communication model with a focus on the application of a numerical approach for computer simulation.

The mathematical model is described by an initial-boundary value problem for a system of parabolic PDEs:

$$\begin{cases} \partial U/\partial t = D_U \Delta U - \gamma_U U - \gamma_{L\to U} LU + F_U(x,U), \\ \partial L/\partial t = D_L \Delta L - \gamma_L L + F_L(x,U),\ 0 \le x \le l,\ 0 \le t \le t_{\mathrm{ob}}, \end{cases} \tag{1}$$

where for the one-dimensional model $U(x,t)$ and $L(x,t)$ are the concentrations of special substances produced by bacteria (N-acyl homoserine lactones, shortly AHL, and Lactonase, respectively) in mol/l, l is linear size of the domain solution in µm, t_{ob} is observation time in h. Model parameters are reported in details in [1] including typical production terms of $F_U(x,U)$ and $F_L(x,U)$ defined with use of normal distribution of bacterial cells as well as Hill's law. A set of boundary conditions as well as initial conditions are imposed to complete the mathematical problem statement:

$$U(0,t) = 0,\ U(l,t) = 0,\ L(0,t) = 0,\ L(l,t) = 0;\ U(x,0) = 0,\ L(x,0) = 0. \tag{2}$$

In order to solve the problem (1)–(2) numerically, we constructed a computational scheme based on implicit finite-difference method. The iterative procedure due to presence of non-linear reaction terms was proposed to be implemented for each time layer. The algorithm for the stochastic process simulation of bacterial nucleation and growth was included into the general scheme. The program application was designed in Matlab to perform series of computational experiments and examine the wave behavior of the time-dependent characteristics for bacterial communication process.

References

[1] Ch. Kuttler, Reaction-diffusion equations and their application on bacterial communication, In: *Handbook of Statistics*, Chapter 4, 55–91 (2017).

[2] A. Meyer, J. A. Megerle, Ch. Kuttler, et al., Dynamic of AHL mediated quorum sensing under flow and non-flow conditions, *Phys. Biol.*, **9**(2), 026007 (2012).

Toda lattice for semi-bounded initial data and classical moment problem

Mikhaylov A.S., **Mikhaylov V.S.**

St. Petersburg Department of V. A. Steklov Institute of Mathematics of the Russian Academy of Sciences, 7, Fontanka, 191023 St. Petersburg, Russia

e-mail: `mikhaylov@pdmi.ras.ru, vsmikhaylov@pdmi.ras.ru`

We define the solution of semi-infinite Toda lattice for wide class of unbounded initial data. By using some ideas of Moser for finite-dimensional case, we derive the evolution of moments of spectral measure of Jacobi operator associated with Toda lattice via the Lax pair.

Acknowledgments. A. S. Mikhaylov and V. S. Mikhaylov were partly supported by RFBR grant № 18-01-00269 and by Volkswagen Foundation project "From Modeling and Analysis to Approximation".

References

[1] A. S. Mikhaylov, V. S. Mikhaylov, Dynamic inverse problem for Jacobi matrices, *Inverse Problems and Imaging*, **13**(3), 431–447 (2019).

[2] A. S. Mikhaylov, V. S. Mikhaylov, Inverse problem for dynamical system associated with Jacobi matrices and classical moment problems, *Journal of Mathematical Analysis and Applications*, **487**(1), (2020).

[3] A. S. Mikhaylov, V. S. Mikhaylov, Finite Toda lattice and classical moment problem, *Nanosystems: Physics, Chemistry, Mathemetics*, **11**(1), 25–29 (2020).

[4] J. Moser, Finitely many mass points on the line under the influence of an exponential potential — an integrable system, *Dynamical Systems, Theory and Applications, Lecture Notes in Physics*, **38**, 67–497 (1975).

[5] M. Toda, *Theory of Nonlinear Lattices*, Springer Series in Solid-State Sciences, **20** (2nd ed.), Springer, Berlin, 1989.

Exchange pulse corresponding to phase synchronism in a flexible plate loaded by gas

Mironov M.A.
Andreyev Acoustics Institute, Shvernik str., 4, Moscow, Russia
e-mail: mironov_ma@mail.ru

Shanin A.V., Korolkov A.I., Kniazeva K.S.
Moscow State University, Leninskie Gory, 1-2, Moscow, Russia
e-mail: korolkov@physics.msu.ru

A problem of excitation of a sonic pulse in a system comprised by an elastic plate and a gas is studied. The source is localized in time and space, so the problem is non-stationary. It is known that such a system can have a phase synchronism point in the spectral domain, i.e. the values of frequency and wavenumber belonging both to the dispersion diagram of the plate and the gas. We demonstrate that such a point leads to appearing of an "exchange pulse" [1], that is a quasi-monochromatic long pulse in the gas.

In the talk, we write down an integral representation of the sonic field and discuss a method of asymptotic evaluation of the 2D Fourier integral. We demonstrate the term responsible for the exchange pulse.

The work is motivated by an experiment made by one of the authors, who recorded sound produced by kicking of a thin (about 3 cm) layer of ice on a pond. The ice plays the role of the plate, and the gas is the air. The experimental signals generally support the theory presented in the talk.

The work is supported by RFBR grant № 19-29-06048.

References

[1] A. V. Shanin, A. I. Korolkov, K. S. Kniazeva, Multi-contour saddle point method on dispersion diagrams for computing transient field components in waveguides, *Proceedings of PIERS 2018*, **90**, 1955–1962 (2018).

On regularization of the Heun functions

Oleg V. Motygin
Institute for Problems in Mechanical Engineering, Russian Academy of Sciences, V.O., Bol'shoy pr. 61, 199178, St. Petersburg, Russia
e-mail: `o.v.motygin@gmail.com`

Heun differential equation was introduced by Karl Heun in 1889 [1]. Its general form is a Fuchsian equation with four regular singular points in the complex z-plane, which are usually chosen to be $z = 0$, 1, a, and ∞. Solutions of the Heun equation generalize many known mathematical functions and appear in many fields of modern physics, see [2, 3].

Evaluation of the functions is based on local power series solutions near the point $z = 0$ derived by the Frobenius method. In [4] a procedure based on power series expansions and analytic continuation is suggested to define solutions in the whole complex plane with branch cuts. However, the Frobenius method generally gives two independent local solutions provided that two roots of the so-called indicial equation are not separated by an integer. For the equation under consideration, the exceptional cases occur at integer values of exponent-related parameter γ of the equation, when one of the local solutions should include a logarithmic term (see [4]). This also means singular behaviour of the Heun functions as functions of γ. In the present work we suggest a method of regularization and redefine Heun functions in some vicinities of the integer values of γ, where the new functions depend C^∞-smoothly on γ.

References

[1] K. Heun, Zur Theorie der Riemann'schen Functionen zweiter Ordnung mit vier Verzweigungspunkten, *Mathematische Annalen*, **33**, 161–179 (1889).

[2] "The Heun project": Heun functions, their generalizations and applications. On-line: `http://theheunproject.org/`.

[3] M. Hortaçsu, Heun functions and their uses in physics, *Mathematical Physics*, 23–39 (2012).

[4] O. V. Motygin, On numerical evaluation of the Heun functions, *Proceedings of Days on Diffraction 2015*, 222–227 (2015).

About the importance of numerical models for quality assessment of guided wave-based structural health monitoring systems

Mueller I.
Bochum University of Applied Sciences, Bochum, Germany
e-mail: `inka.mueller@hs-bochum.de`

Shpak A.N., Golub M.V.
Kuban State University, Krasnodar, Russia
e-mail: `alisashpak7@gmail.com, m-golub@inbox.ru`

Structural Health Monitoring (SHM) systems based on guided waves are nowadays used for the detection, localization and assessment of damages in plate-like structures. Actively introduced waves are interacting with possible damages within the structure, enabling feature extraction of sensed wave signals. Their application in industrial environment is up to now limited. One of the reasons for this restricted application is the lack of a well established method of quality assessment for such SHM systems. The transfer of quality assessment methods from neighbouring methods like ultrasonic testing in non-destructive evaluation proves to be partially possible for a limited number of applications but cannot be transferred to the majority of guided wave-based SHM applications, which monitor larger areas of a structure.

Within this publication, the authors list the current challenges of quality assessment for guided wave-based SHM systems. They present latest research results on how to tackle these challenges. Here they put a special focus on the great opportunities as well as possible difficulties and potential solutions for the usage of simulation of guided waves for the quality assessment of guided wave-based SHM systems.

High contrast approximation for penetrable wedge diffraction

Nethercote M.A., Assier R.C.
School of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL, UK
e-mail: `matthew.nethercote@manchester.ac.uk, raphael.assier@manchester.ac.uk`

Abrahams I.D.
Isaac Newton Institute, University of Cambridge, 20 Clarkson Road, Cambridge, CB3 0EH, UK
e-mail: `i.d.abrahams@newton.ac.uk`

The important open canonical problem of wave diffraction by a penetrable wedge is considered in the high-contrast limit. Mathematically, this means that the contrast parameter, the ratio of a specific material property of the host and the wedge scatterer, is assumed small. The relevant material property depends on the physical context and is different for acoustic and electromagnetic waves for example. Based on this assumption, a new asymptotic iterative scheme is constructed. The solution to the penetrable wedge is written in terms of infinitely many solutions to (possibly inhomogeneous) impenetrable wedge problems. Each impenetrable problem is solved using a combination of the Sommerfeld–Malyuzhinets and Wiener–Hopf techniques.

The resulting approximated solution to the penetrable wedge involves a large number of nested complex integrals and is hence difficult to evaluate numerically. In order to address this issue, a subtle method (combining asymptotics, interpolation and complex analysis) is developed and implemented, leading to a fast and efficient numerical evaluation. This asymptotic scheme is shown to have excellent convergent properties and leads to a clear improvement on extant approaches.

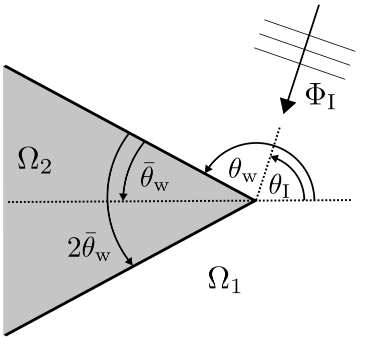


Fig. 1: The geometry of the penetrable wedge problem where Ω_2 (resp. Ω_1) indicates the wedge scatterer (resp. host) region and $\Phi_{\rm I}$ is the incident plane wave.

References

[1] M. A. Nethercote, R. C. Assier, I. D. Abrahams, Analytical methods for perfect wedge diffraction: a review, *Wave Motion*, **93**, 102479 (2020).

[2] M. A. Lyalinov, Diffraction by a highly contrast transparent wedge, *J. Phys. A. Math. Gen.*, **32**(11), 2183–2206 (1999).

[3] A. V. Shanin, On wave excitation in a wedge-shaped region, *Acoust. Phys.*, **42**(5), 612–617 (1996).

[4] V. G. Daniele, The Wiener–Hopf technique for impenetrable wedges having arbitrary aperture angle, *SIAM J. Appl. Math.*, **63**(4), 1442–1460 (2003).

[5] S. P. Groth, D. P. Hewett, S. Langdon, A high frequency boundary element method for scattering by penetrable convex polygons, *Wave Motion*, **78**, 32–53 (2018).

Generation of vector Bessel-type beams via geometrical phase elements for transparent material processing applications

Orlov S., Baltrukonis J., Nacius E., Gotovski P., Ulčinas O.
State Research Institute Center for Physical Sciences and Technology, Industrial Laboratory for Photonic Technologies, Sauletekio ave 3, LT-10222, Vilnius, Lithuania
e-mail: `sergejus.orlovas@ftmc.lt`

Zeroth order Bessel beams are widely used in laser micromachining of transparent materials. The small diameter of central core and elongated focus enables to generate high aspect ratio voids. The simplest way to generate this beam is to induce a conical shape phase with an axicon. However, the quality of the axicon tip is very crucial to generate smooth Bessel beams since it is known that a blunt axicon tip induces large intensity modulation in propagation direction. Alternative Bessel beam generation method is to use a Diffractive Optical Elements (DOEs) that do not suffer from previously mentioned problem [1].

Non-diffracting hugher order Bessel beams and their modified versions are also more widely used nowadays in industry for transparent material micro processing purposes - cutting, drilling etc., due to generation of high aspect ratio micro voids. More and more applications of such beams involve manipulation of their transverse intensity profile andor polarization to create unique tools for novel micro processing applications, for example, asymmetric and multi-peak transverse profiles create directional strain and crack in modified area for glass cutting, while other intensity patters may be used to create complex structures in multiphoton polymerization applications [2].

In this work we start with generation of a zeroth order Bessel beam with Geometric Phase Optical Elements (GPOEs) (manufactured by Workshop of Photonics) acting as a diffractive beam shaping element. Having absolute control of induced beam phase, we have modified mask phase so that half of it had additional phase shift or spatial transposition resulting in creation of fanciful induced beam phase patterns. With the use of laser beam propagation numerical modeling we show that these new phase masks can create various beam transverse intensity patterns such as asymmetrical central core, generation of multiple peaks or even large rings that are highly demanded for various laser micromachining applications. We have chosen couple of most perspective beam shapes and manufactured GPOEs to generate them. The experimentally generated beams were compared to numerical simulations. As the GPOEs are able to work with high power pulses we have also investigated induced transparent material modifications.

Lastly, we demonstrate experimental generation of higher order vector Bessel beams which are notable for their ring-shaped transverse intensity profile together with multi-peak transverse polarization components, where ring diameter and number of peaks in separate polarization components depends on beams order. These unique beams were realized using axicon together with higher order s-plates — spatially variant waveplates based on femtosecond laser written nano gratings in fused silica glass substrates. Induced nanogratings withstands high intensity laser radiation without changing its spatial structure which allows us to use nanograting based elements for ultra-short high-power pulsed laser beam shaping. Generated higher order vector Bessel beams and their separate polarization components were used to inscribe modifications in transparent materials and to investigate beam‘s applicability for ultra-fast laser micro processing purposes.

References

[1] J. Baltrukonis, O. Ulcinas, P. Gotovski, S. Orlov, V. Jukna, Realization of higher order vector Bessel beams for transparent material processing applications, *Proc. SPIE, 11268, Laser-based Micro- and Nanoprocessing XIV*, 112681D (2020).

[2] E. Nacius, P. Gotovski, V. Jukna, O. Ulčinas, T. Gertus, S. Orlov, Generation of Bessel type beams via phase shifted axicons encoded in geometrical phase elements, *Proc. SPIE, 11266, Laser Resonators, Microresonators, and Beam Control XXII*, 112660Z (2020).

Investigation of geometrical phase elements based on chiral and birefringent clusters of nanoparticles using closed-form Mie theory of vector complex source vortices

Orlov, S., Berškys, J., Laurinavičius, K.
State research institute Center for Physical Sciences and Technology, Coherent Optics Laboratory, Sauletekio ave 3, LT-10222, Vilnius, Lithuania
e-mail: sergejus.orlovas@ftmc.lt

Tailoring of functional coatings and thin films to desired applications requires a precise control of both collective and individual attributes of sub-elements of the coating. Usually this can be achieved by the control of the microstructure and its inner workings (architecture) at the nanoscale. Nano-engineering of the clusters of nanoparticles deposited on a glass substrate gives rise to novel optical elements, which are acting as various mode filters, converters of topological charge or of the chirality of the incident beam etc. A good example of such optical elements are the so-called geometrical phase elements (GPE). Their applications are ranging nowadays from devices like metalenses to special optics and also from wavelengths in the ultraviolet to the THz diapason. The reason behind such flexibility is due to variety of different production approaches - lithography based, glancing angle deposition, patterning processes, deposition of sculptured coatings etc. and due to the varying orientation and individual properties of sub-elements of the GPE.

Gaussian beam and higher modes can be introduced by making the position of their source complex (moving it to the complex point). This concept has lead later to the introduction of so-called complex source beams. We build here upon our previous development of complex source vortices (CSV) and introduce an analytical expansion of scalar CSVs into spherical multipoles. We extend it to the vector CSVs and study in detail cases of rather complex polarisation topologies. This expansion enables us to introduce a closed-form analytical Mie theory. We use the analytical Mie theory of vectorial comlex source vortices to study properties of a cluster of three nanoparticles.

Here, we report on a novel approach in the engineering of such GPE's using precisely engineered clusters of nanoparticles. We study in detail optical properties of such nanoclusters and investigate how individual properties of nanoparticles are influencing the collective response of a GPE. As a proof of concept we design a functional coating, which acts as a diffractive optical element for laser beam shaping applications. Lastly we investigate ineraction of such coating with CSVs.

References

[1] S. Orlov, J. Berškys, K. Laurinavičius, Closed-form analytical Mie theory of vector complex source vortices, *Proc. SPIE, 11297, Complex Light and Optical Forces XIV*, 1129705 (2020).

[2] K. Laurinavičius, J. Berškys, S. Orlov, Tailoring response of a cluster of nanoparticles on a substrate and its application for design of geometrical phase elements, *Proc. SPIE, 11289, Photonic and Phononic Properties of Engineered Nanostructures X*, 112891 (2020).

The case of a transformation of degenerating modes typical for a non self-adjoint Schrödinger-type equation

Perel M.V.
Saint Petersburg University, 7/9 Universitetskaya nab., St. Petersburg 199034, Russia
e-mail: m.perel@spbu.ru

The transformation of modes near the points of degeneracy arises in many applied problems. One example is a slowly irregular waveguide, where the phase velocities of the two modes coincides at some distance along the axis of the waveguide or are close to each other. The waves in the waveguide can be of a different nature: acoustic, elastic, or electromagnetic, etc. Other examples come from non-stationary problems of quantum mechanics, where two energy levels approach each other at some point in time. The study of such problems is usually based on the specifics of the applied problems.

We intend to state the problem of degeneracy points in a general form, which will enable us to classify such problems, and to find asymptotic solutions near the points of degeneracy and a transition matrix describing the mode transformation. The problems studied are governed by ordinary differential equations or partial differential equations. Their main feature is that one variable, say x, has a separate meaning compared to others, if any. An example is time in non-stationary quantum problems or coordinate along the axis in the case of waveguides. Another important feature is the presence of a small parameter $\hbar$. Here we investigate the asymptotics of solutions of the equation

$$\hat{\boldsymbol{\mathcal{K}}}(x,\hbar)\boldsymbol{\Psi}(x,\hbar) = -i\hbar\boldsymbol{\Gamma}\frac{\partial\boldsymbol{\Psi}(x,\hbar)}{\partial x}, \qquad \hat{\boldsymbol{\mathcal{K}}}(x,\hbar) \equiv \boldsymbol{\mathcal{K}}(x) + \hbar\boldsymbol{\mathcal{B}}(x), \quad \hbar \ll 1, \tag{1}$$

where $\hat{\boldsymbol{\mathcal{K}}}$, $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\Gamma}$ are self-adjoint operators, $\hat{\boldsymbol{\mathcal{K}}}(x,\hbar) - \hat{\boldsymbol{\mathcal{K}}}(0,\hbar)$, $\boldsymbol{\mathcal{B}}$, $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}^{-1}$ are bounded. The operator $\hat{\boldsymbol{\mathcal{K}}}$ may be a matrix or a differential operator with coefficients dependent on x, which does not contain the derivatives with respect to x. We assume that the spectral problem

$$\boldsymbol{\mathcal{K}}(x)\boldsymbol{\varphi}_j(x) = \beta_j(x)\boldsymbol{\Gamma}\boldsymbol{\varphi}_j(x), \tag{2}$$

has at least two eigenvalues $\beta_j(x)$. The leading term of asymptotics of solutions of (1)

$$\boldsymbol{\Psi}_j = \boldsymbol{\varphi}_j(x)e^{\frac{i}{\hbar}\int^x \beta_j(x')dx'}$$

is expressed in terms of eigenvalues β_j and eigenfunctions $\boldsymbol{\varphi}_j$ of the spectral problem (2). This approach fails if eigenvalues have a degeneracy point $\beta_1(x_*) = \beta_2(x_*) \equiv \beta_*$.

The transformation of modes in the case of a two-dimensional eigenspace corresponding to a double eigenvalue was studied in [1]. Here we study the case typical only for non self-adjoint operators $\boldsymbol{\Gamma}^{-1}\hat{\boldsymbol{\mathcal{K}}}$: the eigenspace corresponding to β_* is one-dimensional; the operator $\boldsymbol{\Gamma}^{-1}\hat{\boldsymbol{\mathcal{K}}}$ can be reduced to the Jordan block in this subspace. The idea for our approach to a non self-adjoint case was taken from [2]. Our results here are a generalization of results obtained for applied problems in [3, 4].

References

[1] I. V. Fialkovsky, M. V. Perel, Mode transformation for a Schrödinger type equation: Avoided and unavoidable level crossings, *J. Math. Phys.*, **61**, 043506 (2020).

[2] K. G. Budden, The critical coupling of modes in a tapered Earth-ionosphere waveguide, *Math. Proc. Cambridge Phil. Soc.*, **77**, 567–580 (1975).

[3] M. V. Perel', Overexcitation of modes in an anisotropic earth-ionosphere waveguide on transequatorial paths in the presence of two close degeneracy points, *Radiophys. Quant. El.*, **33**(11), 882–889 (1990).

[4] M. V. Perel, Asymptotic analysis of partial reflection of modes in slowly inhomogeneous elastic waveguides. *Proc. 8th Int. Conf. on Math. and Num. Aspects of Wave Prop.*, 580–582 (2007).

Broadband sound scattering by intense internal waves

Pereselkov S.A.[1], Rybyanets P.V.[1], Kaznacheeva E.S.[1], Badiey M.[2], Kuzkin V.M.[3]

[1]Voronezh State University, Universitetskaya pl. 1, Voronezh, Russia

[2]University of Delaware, Newark, USA

[3]General Physics Institute of RAS, 38 Vavilova str., Moscow, Russia

e-mail: `pereselkov@yandex.ru, badiey@udel.edu, kumiov@yandex.ru`

The oceanic environment, due to the waveguide dispersion and multimode nature of propagation, has the property of self-organizing of the interference pattern (interferogram) of a broadband source. The stable sets of localized bands is formed in the frequency-distance (time) coordinates.

The interferometric processing of the source field is proposed by based on this mechanism. The proposed interferometric processing implements a coherent accumulation of spectral intensity along localized bands by using of 2D Fourier Transformation. At the output of the integral transformation, the spectral density, which can be called a Fourier Hologram, is localized in focal spots caused by interference of sound field modes of different numbers.

The results of interferometric processing of a full-scale experiment (SWARM'95) of broadband sound scattering on intense internal waves, when they led to horizontal refraction (the first trace) and modes coupling (the second trace) of the acoustic field of the source, are presented. 2D Fourier Transformation of interferograms to holograms registered two non-overlapping localized regions caused by direct and scattered acoustic signals. By filtering these regions and applying inverse 2D Fourier Transformation to them, interferogram corresponding to unperturbed waveguide and interferogram corresponding to hydrodynamic perturbations are reconstructed. This approach allowed us to reconstruct the transfer function of the unperturbed waveguide and the temporary variability of the ocean environment. The velocity of propagation of intense internal waves is estimated by using holograms of the perturbed field. The algorithm for transmitting the unperturbed module of the source spectrum at the background of inhomogeneities of the oceanic environment is described and approved.

The theoretical proofing for this remarkable property of Holograms to transmit unperturbed source images through inhomogeneous random media is based on the representation of the resulting interferogram as a linear superposition of direct and scattered fields. The obtained results may be of interest for underwater communication at presence of uncontrolled inhomogeneities of the ocean environment and for their monitoring. Proposed interferometric processing gives us new understanding of those areas in ocean acoustics where broadband sound scattering can be considered as hydrodynamic perturbations.

The work was supported by RFBR (grants № 19-29-06075 and № 19-38-90326).

An inverse problem for the acoustic wave equation

Pestov L.N.
Immanuel Kant Baltic Federal University, Kaliningrad, Nevskogo 14, 236016, Russia
e-mail: `lpestov@kantiana.ru`

We consider initial boundary value problem for the acoustic wave eqaution

$$\begin{aligned}
&\frac{1}{\rho c^2} u^f_{tt} = \operatorname{div}\Big(\frac{1}{\rho}\nabla u^f\Big) - \sigma u_t \;\text{ in }\; \Omega^{2T} = \Omega \times (0, 2T),\\
&u|_{t=0} = 0,\; u|_{t=0} = 0 \quad \text{in} \quad \Omega,\\
&\frac{1}{\rho} u_\nu|_{\Gamma^T} = f \in F, \quad \Gamma^T = \Gamma \times [0, T], \quad \Gamma = \partial\Omega,
\end{aligned}$$

where positive density ρ and speed of sound c from $C^1(\overline{\Omega})$, $\sigma \in L^\infty(\Omega)$. Control space $F = C_0^\infty(\Gamma^T)$. The solution is u^f.

Inverse problem: to find all coefficients in Ω by the inverse data

$$u^f|_{\Gamma\times[0,2T]}, \quad f \in F \;\text{ and }\; \rho|_\Gamma$$

under assumption of big enough T. More precisely

$$T > 2T^*, \quad T^* = \sup_{x\in\Omega} \operatorname{dist}(x, \Gamma),$$

where distance is understood in the sense of Riemannian metric $|dx|/c(x)$. For solving the inverse problem, we use the Boundary Control Method [1, 2] in the version, that close to the paper [3]. We use also the theory of the Dirichlet-to-Neumann map associated with operator $\operatorname{div}(1/\rho\nabla)$, [4].

References

[1] M. I. Belishev, Equations of Gel'fand–Levitan type in a multidimensional inverse problem for the wave equation, *Mat. Vopr. Teor. Rasprostr. Voln, Part 17, Zap. Nauchn. Sem. LOMI*, **165**, 15–20 (1987); translation in *J. Soviet Math.*, **50**(6), 1940–1944 (1990).

[2] M. I. Belishev, Dynamical systems with boundary control: models and characterization of inverse data, *Inverse Problems*, **17**(4), 659–682 (2001).

[3] L. Pestov, On determining an absorption coefficient and a speed of sound in the wave equation by the BC method, *J. Inv. Ill-Posed Problems*, **22**(2), 245–250 (2014).

[4] J. Sylvester, G. Uhlmann, Global uniqueness for uniqueness for an inverse boundary value problem, *Ann. Math.*, **125**, 153–169 (1987).

Variations of normal modes in shallow-water geoacoustical waveguides caused by the bathymetry perturbation

Petrov P.S., Zakharenko A.D.

V. I. Il'ichev Pacific Oceanological Institute, 43 Baltiyskaya str., Vladivostok, 690041, Russia

e-mail: `petrov@poi.dvo.ru`

Consider a shallow-water environment with the water depth h, sound speed profile $c = c(z)$ and density $\rho = \rho(z)$, where z is the depth variable. Sound field formed in such a waveguide by a source of the frequency f can be represented as a decomposition over the so-called normal modes, i.e., the solutions of the 2D Helmholtz equation of the form $\exp(k_j x)\phi_j(z)$ (x is the horizontal variable). Horizontal wavenumbers k_j and modal functions $\phi_j(z)$ can be obtained by solving acoustical spectral problem of the form [1]

$$\begin{cases} \dfrac{d^2\phi_j}{dz^2} + \dfrac{\omega^2}{(c(z))^2}\phi_j = k_j^2\phi_j\,, \quad z \in]0,h[\,\cup\,]h,H], \\ \phi_j|_{z=0} = 0\,, \\ \phi_j|_{z=H} = 0\,, \\ \phi_j|_{z=h^-} = \phi_j|_{z=h^+}\,, \\ \dfrac{1}{\rho}\dfrac{\mathrm{d}\phi_j}{\mathrm{d}z}\Big|_{z=h^-} = \dfrac{1}{\rho}\dfrac{\mathrm{d}\phi_j}{\mathrm{d}z}\Big|_{z=h^+}\,, \end{cases} \tag{1}$$

where $\omega = 2\pi f$, and H is a sufficiently large value of depth chosen is such a way that we can neglect sound waves propagating below $z = H$. In (1) the second equality expresses the pressure-release condition at the sea surface, and fourth and fifth equalities are continuity conditions for sound pressure and particle velocity at $z = h$, where the functions $c = c(z)$ and $\rho = \rho(z)$ have a finite-jump discontinuity (the superscripts '+' and '−' signify the values above and below the interface, respectively).

In realistic models of shallow-water waveguides water depth is normally a function of one or two horizontal variables, i.e. $h = h(x)$ or $h = h(x, y)$, and it is important to take the dependence of k_j and $\phi_j(z)$ on h into account. In fact, k_j is usually a smooth function of h, and therefore we can consider an expansion

$$k_j(h) = k_j(h_0) + \frac{\partial k_j}{\partial h}\Big|_{h=h_0}\Delta h + \frac{\partial^2 k_j}{\partial h^2}\Big|_{h=h_0}\frac{\Delta h^2}{2!} + \dots\,, \tag{2}$$

where $k_j(h)$ is perturbed value of the modal wavenumber, and $\Delta h = h - h_0$ is the water depth perturbation. Equation (2) can be considered a second-order perturbation theory for the Sturm–Liouville operator in (1), where the perturbation is caused by the change in the water-bottom interface depth.

In our work we derive explicit formulae for $\frac{\partial k_j}{\partial h}$ and $\frac{\partial^2 k_j}{\partial h^2}$ and the respective derivatives of the mode functions $\phi_j(z)$. We also investigate the accuracy of the approximation (2) and discuss its applications in 3D models of sound propagation in shallow water.

The reported study was supported by the Russian Foundation for Basic Research under the contracts № 18-05-00057_a and № 18-35-20081_mol_a_ved.

References

[1] F. Jensen, W. Kuperman, M. Porter, H. Schmidt, *Computational Ocean Acoustics*, Springer, New York, et al., 2011.

Diffraction by a grid

Philippov V.B.
St. Petersburg Branch of Mathematical Institute RAS, St. Petersburg, Russia
e-mail: `filippov@pdmi.ras.ru`

The problem of diffraction of a plane wave $u^{\mathrm{inc}} = \exp\big(ik(x\alpha_0 + z\sqrt{1-\alpha_0^2})\big)$ by the periodic grid

$$S = \bigcup_{n=-\infty}^{\infty} S_n,$$

$$S_n = \{(x,z) : z = H, nL \le x < nL + a; z = 0, nL + a \le x < (n+1)L\},$$

is considered.

For two-dimensional Dirichlet problem, the current $\psi()$ at the boundary satisfies the following integral equation:

$$u^{\mathrm{inc}}(\mathbf{x}) = \frac{i}{4}\int_S H_0^1(kR)\psi(\mathbf{x}')dS$$

where $R = |\mathbf{x} - \mathbf{x}'|$, $\mathbf{x}, \mathbf{x}' \in S$.

We will search for the current in the form of a quasi-periodic function

$$\psi(x+L, z) = \exp(ikL\alpha_0)\psi(x,z).$$

Then the problem is reduced to solving the integral equation on one segment borders:

$$u^{\mathrm{inc}}(\mathbf{x}) = \int_{S_0} K(\mathbf{x},\mathbf{x}')\psi(\mathbf{x}')dS$$

where

$$K(\mathbf{x},\mathbf{x}') = \frac{i}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(ik((x-x')\alpha + |z-z'|\sqrt{1-\alpha^2}))}{1-\exp(ikL(\alpha-\alpha_0))}\frac{d\alpha}{\sqrt{1-\alpha^2}}.$$

The integral equation is reduced to a system of algebraic equations using the method proposed by the author at last year's DD session. Analysis of the accuracy dependence of the solution by the crushing grade is conducted.

The case of a finite grid is also considered. The issue of influence of the size of the object on the scattering diagram, as well as the estimation Kirchhoff approximation accuracy and the influence of the deviation of the shape of the boundary from perfect is discussed.

Paraxial beams and related solutions of the Helmholtz equation

Alexandr B. Plachenov
MIREA – Russian Technological University, Moscow, Russia
e-mail: a_plachenov@mail.ru

Aleksei P. Kiselev
St. Petersburg Department of V. A. Steklov Institute of Mathematics, Russia;
St. Petersburg State University, Russia;
Institute for Problems in Mechanical Engineering RAS, St. Petersburg, Russia
e-mail: aleksei.kiselev@gmail.com

Since the beginning of the 1960s, a high-frequency asymptotic construction of paraxial beams solutions of the Helmholtz equation

$$u_{xx} + u_{yy} + u_{zz} + k^2 u = 0, \quad k > 0, \tag{1}$$

localized near the z-axis has been known. It is based on approximate replacement of u by

$$\widetilde{u} = e^{ikz} W, \tag{2}$$

where W satisfies the Leontovich "parabolic equation"

$$2ikW_z + W_{xx} + W_{yy} = 0. \tag{3}$$

Solutions of (3) with a Gaussian-type localization near the z-axis were addressed in countless publications (see, e.g., [1, 2]). In the early 1970s, the question has arisen of the existence of exact solutions of equation (1) having the same high-frequency asymptotic behavior. Attempts to explicitly construct them do not stop (see, e.g., some references to earlier research in [3]). Such solutions satisfying (1) in the free space and not involving backward propagating waves are known yet only as complicated superpositions of plane waves (see [3–6]). We generalize these results without relying on the assumption of a Gaussian-type localization. Our analysis is applicable, e.g., to solutions introduced in [7].

References

[1] V. M. Babich, V. S. Buldyrev, *Asymptotic Methods in Short-Wavelength Diffraction Theory*, Alpha Science, Oxford, 2007.

[2] A. P. Kiselev, Localized light waves: Paraxial and exact solutions of the wave equation (a review), *Opt. Spectr*, **102**, 603–622 (2007).

[3] A. P. Kiselev, Time-harmonic Gaussian beams: Exact solutions of the Helmholtz equation in free space, *Opt. Spectr.*, **123**, 935–939 (2017).

[4] J. Lekner, Electromagnetic pulses, localized and causal, *Proc. Roy. Soc. London A*, **474**(2209), pap. 20170655 (2018).

[5] A. P. Kiselev, A. B. Plachenov, Astigmatic Gaussian beam: Exact solution of the Helmholtz equation, *2018 Days on Diffraction*, 168–170 (2018).

[6] A. P. Kiselev, A. B. Plachenov, Astigmatic Gaussian beams: Exact solutions of the Helmholtz equation in free space, *J. Phys. Commun.*, Vol. **3**, pap. 11504 (2019).

[7] E. Abramochkin, E. Razueva, Product of three Airy beams, *Opt. Lett.*, **36**, 3732–3734 (2011).

Radiation pattern of borehole GPR slot antenna

Popov A.V.[1], Garbatsevich V.A.[1], Morozov P.A.[1,2], Morozov F.P.[2], Prokopovich I.V.[1]
[1]Pushkov Institute of Terrestrial Magnetism, Ionosphere and Radio Wave Propagation, Troitsk, Moscow, Russia
[2]JSC Company VNIISMI, Moscow, Russia
e-mail: popov@izmiran.ru, lozaberk@yandex.ru

Ground penetrating radar (GPR) finds a vast field of application, including borehole EM probing. One of the problems arising is the design of antennas with azimuthal directivity [1]. We suggest a solution that consists in the use of a cylindrical transmitter antenna with a longitudinal slit fed by a pulsed current source. Angular dependence of the emitted radar pulse is achieved due to creeping wave attenuation in the shadow region. Similar diffraction effects assure directivity of the receiver antenna. For a rough estimate of the pulsed GPR directivity, we study the model problem of harmonic wave radiation at the pulse central frequency. The obtained analytical solution demonstrates a pronounced radiation pattern for practical values of operating frequency and antenna dimensions. In our talk, along with a sketch of theory, we give an account of the proposed borehole slot antenna first field tests. The prototypes of the transmitter and receiver antennas were made of thin-wall steel pipes enclosing commercial Loza-V GPR units [2]. Our experiments confirm feasibility of the proposed solution.

Consider an infinite cylindrical antenna with a longitudinal slit $\varphi < \alpha = d/a$, excited by a harmonic potential $2Ue^{-i\omega t}$ and immersed in a homogeneous dielectric medium $\varepsilon > 1$; see Fig. 1. We calculate wave field distribution at distances comparable to the cylinder diameter. In contrast to rigorous theory [3], the method of successive approximations yields simple analytical formulas. All EM field components can be expressed via $H_z = H(r, \varphi)$ satisfying the Helmholtz equation

$$\frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial H}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H}{\partial \varphi^2} + k^2 \varepsilon(r)\, H = 0, \quad \varepsilon(r) = \begin{cases} \varepsilon, & r > a \\ 1, & r < a \end{cases}. \tag{1}$$

with boundary conditions on the metal antenna wall and continuity of the tangential field components requirement in the slit sector:

$$\begin{aligned} &(a):\ \left(\frac{\partial \overset{+}{H}}{\partial r}\right) = \left(\frac{\partial \overset{-}{H}}{\partial r}\right), \quad (b):\ \overset{+}{H} - \overset{-}{H} = -\frac{4\pi}{c} I(\varphi)\ , \quad |\varphi| > \alpha; \\ &(c):\ \frac{1}{\varepsilon}\left(\frac{\partial \overset{+}{H}}{\partial r}\right) = \left(\frac{\partial \overset{-}{H}}{\partial r}\right), \quad (d):\ \overset{+}{H} = \overset{-}{H}\ , \quad |\varphi| < \alpha \end{aligned} \tag{2}$$

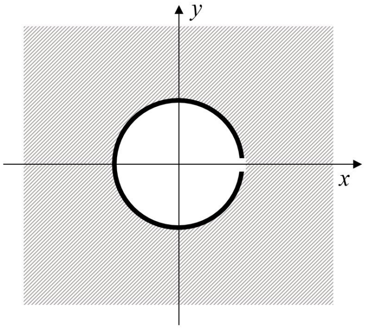


Fig. 1: Schematic plot of a slot borehole antenna.

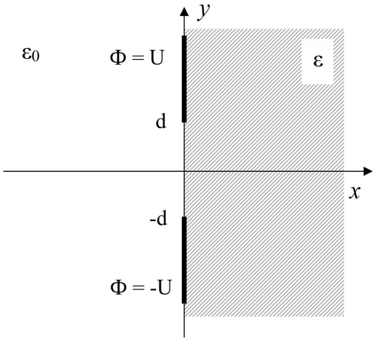


Fig. 2: Geometry of the narrow gap.

Here, $I(\varphi)$ is surface current density in the conductive wall $r = a$. In addition, the solution of Eq. (1) must be regular at $r = 0$ and satisfy the radiation condition at infinity. These requirements uniquely determine the wave field $H(r, \varphi)$ for a given current $I(\varphi)$.

The excited wave field inside the cylinder and in the surrounding medium can be represented as Fourier series

$$H(r,\varphi)=\sum_{m=-\infty}^{\infty} X_m\,Y_m(r)\,e^{im\varphi};\quad Y_m(r)=\left\{\frac{J_m(kr)}{J'_m(ka)},\ r<a\,;\ \ \sqrt{\varepsilon}\frac{H_m^{(1)}(kr\sqrt{\varepsilon})}{H_m^{(1)\prime}(ka\sqrt{\varepsilon})}\ ,\ r>a\right\}. \tag{3}$$

automatically satisfying the contact condition (2c). The other conditions yield an overdetermined equation set

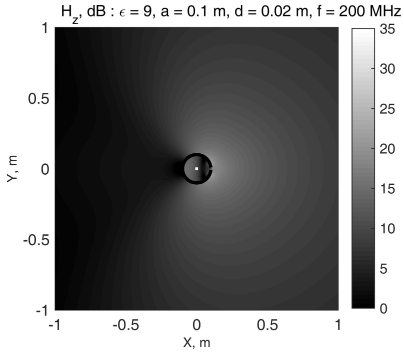
$$\sum_m X_m e^{im\varphi}=0,\quad |\varphi|>\alpha;\quad \sum_m A_m X_m e^{im\varphi}=\begin{cases}-\frac{4\pi}{c}I(\varphi), & |\varphi|>\alpha\\ 0, & |\varphi|<\alpha\end{cases}$$
$$\text{with}\ \ A_m=\sqrt{\varepsilon}\frac{H_m^{(1)}(ka\,\sqrt{\varepsilon})}{H_m^{(1)\prime}(ka\,\sqrt{\varepsilon})}-\frac{J_m(ka)}{J'_m(ka)}. \tag{4}$$

In a strict formulation, determination of the Fourier coefficients X_m from the infinite system of equations (4) is similar to the Riemann–Hilbert problem of the theory of analytic functions [3]. An elementary solution can be found by the perturbation method. Following Sommerfeld [4], consider the case of a narrow gap $\alpha \ll 1$. Obviously, the screen curvature can be neglected (see Fig. 2), and for $d \ll \lambda$ the wave field in the gap vicinity can be calculated in quasi-static approximation: $\vec{E}=-\operatorname{grad}\Phi(x,y)$, $\Delta\Phi=0$. Boundary conditions for harmonic function $\Phi(x,y)$ are: given wall potentials $\Phi(0,y)=\pm U$, $|y|>d$, and contact boundary conditions: $\overset{+}{\Phi}(0,y)=\overset{-}{\Phi}(0,y)$, $\varepsilon\overset{+}{\Phi}_x(0,y)=\overset{-}{\Phi}_x(0,y)$, $|y|<d$. Spatial potential distribution is determined as a function of complex variable $z=x+iy$: $\Phi(x,y)=\frac{2}{\pi}U\,\mathrm{Im}\big(\mathrm{Arsinh}\,\frac{z-a}{d}\big)$, whence

$$E_\varphi(a,\varphi)=-i\sum_{-\infty}^{\infty}X_m\,e^{im\varphi}\approx\frac{2U}{\pi\sqrt{d^2-a^2\varphi^2}},\quad |\varphi|<\alpha. \tag{5}$$

From here, by successive approximations, we find the Fourier coefficients

$$X_m^{(0)}=\frac{i}{2\pi}\int_{-\pi}^{\pi}E_\varphi(a,\varphi)\,e^{-im\varphi}d\varphi=\frac{iU}{\pi a}J_0(m\alpha),$$
$$X_m^{(1)}=\left(1-\frac{\alpha}{\pi}\right)X_m^{(0)}-\frac{1}{\pi A_m}\sum_{n\neq m}A_n X_n^{(0)}\frac{\sin(n-m)\,\alpha}{n-m}. \tag{6}$$

(a)


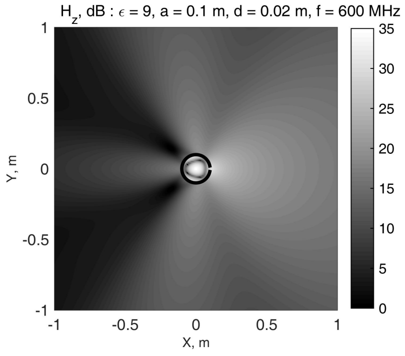
(b)


Fig. 3: Radiation pattern for $f=200$ MHz (a) and $f=600$ MHz (b); tube diameter 20 cm.

Substitution of the refined coefficients (6) into the Fourier series (3) provides approximate fulfillment of the boundary conditions and quantitative description, with practical accuracy, of the antenna radiation dependence on frequency, medium dielectric permittivity, cylinder diameter, and gap width. The main physical phenomena that determine the radiation pattern of a tubular slot antenna are diffraction of the emitted wave on the outer surface of the cylinder and resonant field amplification in the inner cavity at selected frequencies. The calculated wave field patterns exhibit pronounced radiation directivity and frequency dependence (Fig. 3) that must be taken into account when designing a pulsed borehole GPR.

The authors dedicate this publication to the memory of V. V. Kopeikin who proposed the idea of using slot antennas for borehole GPR. We are grateful to D. E. Edemskij for useful discussions. This work was supported by the Russian Foundation for Basic Research, grant № 18-02-00185.

References

[1] K. W. A. van Dongen, R. van Vaard, S. van der Baan, P. M. van den Berg, J. T. Fokkema, A directional borehole radar system, *Subsurface Sensing Technologies and Applications*, **3**, 327–346 (2002).

[2] V. V. Kopeikin, I. V. Krasheninnikov, P. A. Morozov, A. V. Popov, G. Y. Fang, X. Liu, X. Zhou, Experimental verification of LOZA-V GPR penetration depth and signal quality, *Proc. 4th Int. Workshop on Advanced GPR*, 230–233 (2007).

[3] V. P. Shestopalov, *Method of Riemann–Hilbert Problem in Diffraction and Wave Propagation Theory*, University Press, Kharkov, Ukraine, 1971 (in Russian).

[4] A. Sommerfeld, *Partial Differential Equations in Physics*, Academic Press, N.Y., 1964.

GPR probing of smoothly layered subsurface medium: 3D analytical model

Popov A.[1], Prokopovich I.[1], Morozov P.[1], **Marciniak M.**[2]
[1]Pushkov Institute of Terrestrial Magnetism, Ionosphere and Radio Wave Propagation, Troitsk, Moscow, Russia
[2]National Institute of Telecommunications, Warsaw, Poland
e-mail: popov@izmiran.ru, m.marciniak@itl.waw.pl

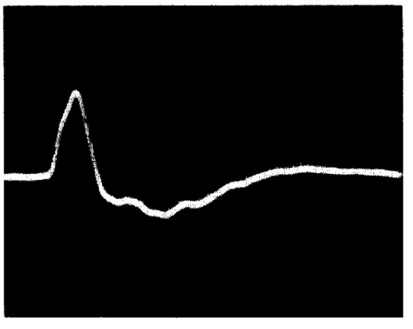
Despite the growing possibilities of computers and modern numerical techniques, the problem of subsurface sensing with a ground penetrating radar (GPR) requires the development of analytical approaches revealing fundamental laws of electromagnetic pulse propagation in non-uniform natural environments and simplifying the solution of direct and inverse problems. While direct wave propagation through material environments with gradually changing parameters is successfully described by geometric optics or Wentzel–Kramers–Brillouin (WKB) method of quantum mechanics, but backward reflected waves in such medium can be exponentially small and lie beyond the accuracy of these methods. In that respect, coupled-wave WKB method describing partial reflections from smooth permittivity gradients looks particularly promising.

An analytical technique, based on the coupled-wave WKB approximation transformed to time domain, has been developed for 1D case in [1]. In this case a non-uniform half-space $z > 0$ is characterized by a real-valued relative permittivity profile $\varepsilon(z)$ and a vacuum magnetic permeability μ_0. The source is placed at $z = 0$ and transient field $E(s, z)$ is generated by known probing pulse $f(t)$ which defines a non-homogeneous boundary condition at $z = 0$. This method allows one to write down an integral equation for the reflected pulse $g(t)$ — the half-space response to the input electromagnetic pulse. Moreover, such a statement yields the 1D inverse problem solution $\varepsilon(z)$ for a given probing pulse $f(t)$ and measured reflected pulse $g(t)$.

Further development of the coupled-wave WKB approach [2] includes more difficult 1.5-D scenarios when the medium is still assumed to be horizontally stratified, but the transmitted source is described by a current line stretched along the air-ground interface, producing a two-dimensional (2D) transient electromagnetic field. In this geometry we can take into account spacing between transmitter and receiver antennas. We still assume here a uniform current distribution along the thin wire antenna. The Fourier–Laplace transform allows one to reduce the time-domain boundary value problem to an ordinary differential equation, which is similar to 1D case. This semi-analytical approach shows good results in comparison with accurate FDTD solutions for different medium parameters and antenna offsets.

In this work, we extend our results on more realistic scenarios, taking into account the finite length and non-uniform current distribution in the transmitter GPR antenna. To finish up the obtained solution, one has to derive current distribution in the antenna lying on the ground surface, which implies solving an integro-differential equation. In order to avoid excessive computational difficulties, we propose a physically justified analytic model following from the results of our GPR experiments [3]. Figure 1 illustrates qualitative agreement of this model with the pulse waveform in a resistively loaded dipole antenna measured in classical experiments [4].

(a)



(b)

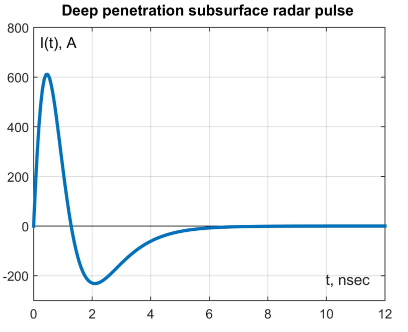


Fig. 1: a) Measured current pulse in a resistively loaded GPR antenna [4]; b) model GPR pulse [3].

This work has been supported in part by RFBR grant № 18-02-00185.

References

[1] V. A. Vinogradov, V. V. Kopeikin, A. V. Popov, An approximate solution of 1D inverse problem, *Proceedings of the 10th International Conference on Ground Penetrating Radar*, 95–98 (2004).

[2] I. Prokopovich, A. Popov, L. Pajewski, M. Marciniak, Application of coupled-wave Wentzel–Kramers–Brillouin approximation to ground penetrating radar, *Remote Sens.*, **10**(1), 22 (2018).

[3] A. I. Berkut, D. E. Edemsky, V. V. Kopeikin, P. A. Morozov, I. V. Prokopovich, A. V. Popov, Deep penetration subsurface radar: Hardware, results, interpretation, *Proceedings of the 9th International Workshop on Advanced Ground Penetrating Radar*, 1–6 (2017).

[4] S. Arcone, Numerical studies of the radiation patterns of resistively loaded dipoles, *Journ. Applied Geophysics*, **33**, 39–52 (1995).

On matching of integral representation for whispering gallery wave propagating along smooth surface in $\mathbb{R}^3$ and a source of the wavefield

Popov M.M.
St. Petersburg Department of the Steklov Mathematical Institute, Fontanka emb. 27, St. Petersburg, 191023, Russia
e-mail: mpopov@pdmi.ras.ru

In the previous paper [1], an integral representation of whispering gallery waves was proposed. It consists in the following.

Denote by the vector function $\vec{r}(s,\gamma)$ in $\mathbb{R}^3$ a flow of geodesic lines on the surface associated with the whispering gallery wave propagating on the surface (s is arc length of geodesic line, γ is a parameter specifying the line).

For each geodesic line from $\vec{r}(s,\gamma)$ a specific asymptotic solution of wave equation is constructed: it is localized in a narrow vicinity of the geodesic line and remains free of any singularity on caustics along the line. The global wavefield of the whispering gallery is presented by superposition (or integral over γ) of these solutions. Thus at this point it resembles the method of Gaussian beam summation method [2, 3].

Present report is devoted to a problem of matching of the integral representation for the whispering gallery wave and a source generating this wave.

References

[1] M. M. Popov, On theory of surface wave propagation on smooth strictly convex surfaces embedded in $\mathbb{R}^3$, *Proceedings of the International Conference DAYS on DIFFRACTION 2019*, 159–162 (2019).

[2] M. M. Popov, A new method of computing wave fields in the high-frequency approximation, *J. Soviet Math.*, **20**(1), 1869–1882 (1982).

[3] M. M. Popov, N. M. Semtchenok, P. M. Popov, A. R. Verdel, Depth migration by the Gaussian beam summation method, *Geophysics*, **75**(2), 1MA–Z39 (2010).

Non-stationary excitation of a trapped mode in a string on an elastic foundation with a moving linear oscillator

Poroshin I.O.[1], Gavrilov S.N.[1,2], Shishkina E.V.[2]

[1]Peter the Great St. Petersburg Polytechnic University (SPbPU), St. Petersburg, Russia

[2]Institute for Problems in Mechanical Engineering RAS, St. Petersburg, Russia

e-mail: poroshin_io@mail.ru, serge@pdmi.ras.ru, shishkina_k@mail.ru

We consider unsteady oscillations of an infinite string on an elastic foundation with a non-uniformly moving discrete oscillator. The governing equations are

$$u'' - \ddot{u} - u = -P(t)\delta\big(x - \ell(t)\big), \tag{1}$$

$$M u(\ell(t), t)^{\cdot\cdot} = -P(t) + p(t) - K u(\ell(t), t). \tag{2}$$

In such a system, in the case of a constant speed of the oscillator, a trapped mode can exist under certain conditions. Applying an unsteady external excitation to this system leads to the emergence of string vibrations localized near the inhomogeneity. Earlier, Gavrilov and Indeitsev solved a similar problem for a moving pure inertial inclusion [1].

In the case of a non-uniform motion of the inclusion, using successively the method of stationary phase and the method of multiple scales, we show that the expression for the amplitude of localized oscillation is

$$W = C_0 \sqrt{\frac{M\Omega_0^2 - K}{\Omega_0(M^2\Omega_0^2 - KM + 2)}}, \tag{3}$$

where C_0 is an arbitrary constant, Ω_0 is the trapped mode frequency.

The results are verified by independent numerical calculations based on the solution of the Volterra integral equation of the second kind. We got a good agreement between the analytical and numerical solutions.

References

[1] S. N. Gavrilov, D. A. Indeitsev, The evolution of a trapped mode of oscillations in a “string on an elastic foundation — moving inertial inclusion” system, *PMM J. Appl. Math. Mech.*, **66**, 825–833 (2002).

Asymptotic solution of the Cauchy problem for the wave equation with fast oscillating coefficient and localized initial conditions

Sergeev S.A.
Ishlinsky Institute for Problems in Mechanics RAS, Moscow, Russia
e-mail: `sergeevse1@yandex.ru`

We consider the Cauchy problem for the wave equation with fast oscillating coefficient

$$\frac{\partial^2}{\partial t^2}u = C^2\left(\frac{\Theta(x)}{\varepsilon},\, x\right)\Delta u, \quad x \in \mathbb{R}^n, \qquad u|_{t=0} = V\left(\frac{x}{\mu}\right), \qquad u_t|_{t=0} = 0.$$

Such equation arises, for example, in the underwater acoustic problems for the sound propagation. The initial function $V(x/\mu)$ is localized near point $x = 0$ and $0 < \mu \ll 1$ is the small parameter. Vector-function $\Theta(x) = (\theta_1(x), \ldots, \theta_m(x))$, $m \le n$, where $\theta_k(x)$ are smooth real-value functions and gradients $\nabla\theta_k(x)$ are linear independent for all x. Function $C^2(y, x)$, $y \in \mathbb{R}^m$ is 2π-periodical with respect to each variable y_j and also smooth with respect to all variables. Also the following condition holds $0 < c_m \le C^2(y, x) \le c_M$. Parameter ε is also small: $0 < \varepsilon \ll 1$.

We need to construct the asymptotic solution for the initial problem. It is obvious such asymptotic depends on the ratio ε/μ.

Using the adiabatic approximation in the operator form, we reduce the initial equation to the homogenized equation with smooth coefficients. Using the technique of the modified Maslov canonical operator, we construct the main part of the asymptotic solution for the Cauchy problem for this equation.

Depending on the ratio ε/μ we can replace the homogenized equation by another one, which has simplier form. If $\varepsilon \sim \mu^{3/2}$ then such equation has the form of the linear Boussinesq equation. If $\varepsilon \sim \mu^{\alpha}$, $\alpha > 3/2$, then we can consider the wave equation. The main parts of the asymptotic solutions for these equations and the homogenized equation are the same.

This work was supported by the Government Budget theme № AAAA-A17-117021310377-1 and RFBR grant № 18-31-00148 mol_a.

Representation of Green's functions as integrals on dispersion diagrams

Shanin A.V., **Korolkov A.I.**, Makarov O.I.
Moscow State University, Leninskie Gory, 1-2, Moscow, Russia
e-mail: `korolkov@physics.msu.ru`

Several 2D problems of finding Green's function are studied in the talk. They are the problems on square and triangular lattices, and model vector problems in anisotropic materials. In all cases, the Green's function can be found in the form of 2D Fourier integrals and then rewritten as plane wave decompositions. The dispersion diagrams, that are the sets of all (possibly complex) waves in the system, play an important role in building the representations.

In the talk we demonstrate that the dispersion diagram is a complex manifold in the space of two complex wavenumbers. Thus, the dispersion diagram is smooth everywhere, and it can be equipped with a complex structure. The plane wave decomposition can be treated as a contour integral of some analytic 1-form on the dispersion diagram.

The interpretation of plane decompositions of Green's functions as contour integrals on the dispersion diagram are useful in building the Sommerfeld integrals for more complicated diffraction problems [1].

The work is supported by the RFBR grant № 19-29-06048.

References

[1] A. V. Shanin, A. I. Korolkov, Sommerfeld-type integrals for discrete diffraction problems, *arXiv*: 1908.04764v2, (2019).

Complex WKB method for difference Dirac equation

Shchetka E.
St. Petersburg Department of Steklov Mathematical Institute of the Russian Academy of Sciences, 27 Fontanka, St. Petersburg, 191023, Russia;
Chebyshev Laboratory, St. Petersburg State University, 14th Line, 29B, Vasilyevsky Island, St. Petersburg, 199178, Russia
e-mail: e.shchetka@spbu.ru

We study analytic solutions to the difference Dirac equation

$$\begin{pmatrix} mc^2 + v(z) & cd^* \\ cd & -mc^2 + v(z) \end{pmatrix} \begin{pmatrix} u_1(z) \\ u_2(z) \end{pmatrix} = E \begin{pmatrix} u_1(z) \\ u_2(z) \end{pmatrix}, \quad z \in \mathbb{C}, \tag{1}$$

where $m, c \geq 0$ are parameters, $(du)(z) = u(z+h) - u(z)$ with $h > 0$, v is an analytic function, and $E \in \mathbb{C}$ is a spectral parameter.

As $h \to 0$, we describe the asymptotics of the analytic solutions to (1) in the framework of the complex WKB method for different classes of potentials v. We consider the following three classes:

- functions analytic in a bounded domain in the complex plane;
- trigonometric polynomials;
- functions analytic in a neighborhood of $\mathbb{R}$ and decaying as $|z|^{-1-\tau}$ for a fixed $\tau > 0$ ($|z| \to \infty$).

As a matrix problem, the asymptotic formulae for the solutions contain the Berry phase that was discussed in [1].

This work was supported by RFBR grant № 17-51-150008.

References

[1] A. Fedotov, E. Shchetka, Difference equations in the complex plane: quasiclassical asymptotics and Berry phase, to appear in *Applicable Analysis (GAPA)*, 2020.

3D mathematical model for the simulation of piezo-induced guided waves in an elongated plate-like structure

Shpak A.N., Golub M.V., Glinkova S.A., Khanazaryan A.D.
Institute for Mathematics, Mechanics and Informatics, Kuban State University, Krasnodar, 350040, Russian Federation
e-mail: alisashpak7@gmail.com, m_golub@inbox.ru

Structural health monitoring (SHM) is a field combining methods and applications, intended for continuous monitoring of a health state of a structure and estimation of a remained life-cycle due to revealed damage. Acousto-ultrasonic SHM methods based on usage of the guided waves are nowadays studied thoroughly and considered to be effective and reliable. These methods usually employ surface mounted or in-built piezoelectric transducers for guided waves excitation and sensing. Still, before the broad implementation of the guided wave-based SHM systems into real-life objects, a number of theoretical problems are to be solved. For this purpose, effective and reliable mathematical models simulating guided waves excitation, propagation and scattering in elongated structures are necessary.

A semi-analytical hybrid approach for simulation of the guided waves excitation by a partially debonded piezoelectric transducer in the two-dimensional case was presented in [1]. The model allows calculating complex-valued eigenfrequencies and analysing characteristics describing elastic waves propagation in a layered structure for the different health state of transducers. It was shown in [2] that 2D model is limited, and its accuracy is worse at higher frequencies. Additionally, some physical values cannot be calculated within the 2D model, because they depend, for example, on the total electroded area. Therefore, it is essential to develop a full 3D coupled mathematical model for simulation of the piezo-induced wave-field in a layered structure.

In this paper, we present a 3D mathematical model for the simulation of a dynamic interaction of a piezoelectric transducer and an elastic waveguide. For this purpose, the mathematical model, obtained in [1], is enhanced to the three-dimensional case. The piezoelectric structure is modelled using the frequency domain spectral element method, which is effective for simulation of dynamics of a complex-shaped transducer. The elastic wave propagation in a layer is modelled by the semi-analytical boundary integral equation method [3], which allows simulation of the separate Lamb wave modes propagation and calculation of the wave energy, transferred from the transducer to the structure. The coupling of these two methods is based on the continuity of displacements and stresses in the contact area. Employment of the collocation method and Galerkin method together with the first order splines and axis-symmetric [3] functions for traction vector interpolation is discussed. The convergence is analysed and the developed method is compared with the standard FEM software.

The research is supported by the grants of the President of the Russian Federation for state support of young Russian scientists (project № MK-470.2020.1) and the Russian Foundation for Basic Research (project № 18-501-12069).

References

[1] M. V. Golub, A. N. Shpak, Semi-analytical hybrid approach for the simulation of layered waveguide with a partially debonded piezoelectric structure, *Applied Mathematical Modelling*, **65**, 234–255 (2019).

[2] A. N. Shpak, I. Mueller, M. V. Golub, C.-P. Fritzen, Theoretical and experimental investigation of Lamb waves excited by partially debonded piezoelectric transducers, *Smart Materials and Structures* (accepted), doi: 10.1088/1361-665X/ab75a1.

[3] E. V. Glushkov, N. V. Glushkova, On the efficient implementation of the integral equation method in elastodynamics, *Journal of Computational Acoustics*, **9**(3), 889–898 (2001).

Localized oscillation in a linear mass-spring chain on an elastic foundation

Shvarev N.G.[1], Shishkina E.V.[2], Gavrilov S.N.[1,2]

[1]Peter the Great St. Petersburg Polytechnic University (SPbPU), St. Petersburg, Russia

[2]Institute for Problems in Mechanical Engineering RAS, St. Petersburg, Russia

e-mail: `shvarev_ng@spbstu.ru, shishkina_k@mail.ru, serge@pdmi.ras.ru`

We obtain the necessary and sufficient conditions for existence of localized natural oscillation in an infinite linear mass-spring chain with nearest-neighbor interactions lying on an elastic foundation. It is assumed that the chain has spatially uniform properties everywhere except of a single defect. The governing equations are

$$\ddot{u}_n = (u_{n+1} - 2u_n + u_{n-1}) - \alpha u_n, \qquad n \in \mathbb{Z}, \quad n \neq 0, \quad \alpha > 0; \tag{1}$$

$$m\ddot{u}_0 = (u_1 - 2u_0 + u_{-1}) - \beta u_0, \quad m \geq 0, \quad \beta \lesseqgtr 0. \tag{2}$$

Some particular cases of this spectral problem were considered before in [1].

References

[1] I. V. Andrianov, V. V. Danishevs'kyy, A. L. Kalamkarov, Vibration localization in one-dimensional linear and nonlinear lattices: discrete and continuum models, *Nonlinear Dynamics*, **72**, 37–48 (2013).

Synoptic eddy influence on the accuracy of the solution of the acoustic ranging problems

Sorokin M.A.
Far Eastern Federal University, 8 Sukhanova str., Vladivostok, 690950, Russia
e-mail: `sorokin.ma@students.dvfu.ru`

Petrov P.S., Kaplunenko D.D., Stepanov D.V.
V. I. Il'ichev Pacific Oceanological Institute, 43 Baltiyskaya str., Vladivostok, 690041, Russia
e-mail: `petrov@poi.dvo.ru`

Acoustical ranging problems have important practical applications in underwater acoustics. Their accurate solution is crucial for the positioning of underwater vehicles. However, various inhomogeneities of the bottom relief and sound speed in the water column can affect the accuracy of acoustical ranging [1].

The goal of the presented study is to qualitatively and quantitatively describe an error in the range estimation caused by the presence of a synoptic eddy on the acoustical path. To this end, we estimated the influence of the eddy on modal group velocities [2].

Consider as an example a 300 km long acoustical path in the sea of Japan (see Fig. 1) and assume that a source of navigation signals (SNS) is located on the shelf and that it transmits pulse signals with central frequency of 400 Hz.

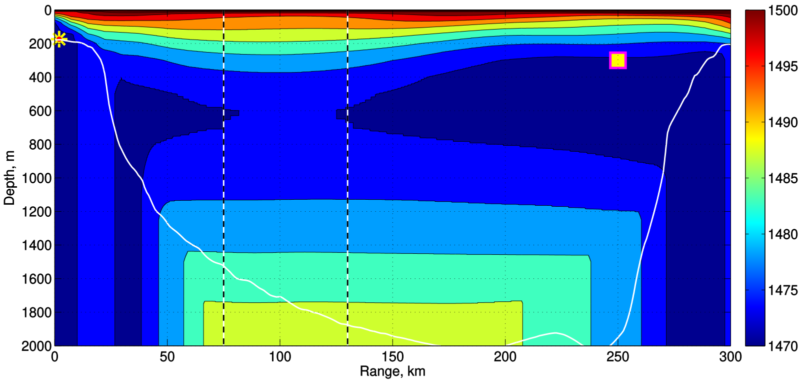


Fig. 1: A 2D waveguide in the Sea of Japan with a synoptic eddy localized between $r = 75$ km and $r = 130$ km (see vertical lines). A contour plot represents sound speed field in the water column. White solid line shows bathymetry. SNS is depicted by a circle (located on the shelf), and the receiver is shown by a square.

Arrival time of the signals at the reception point are determined by the variation of modal group velocities $v_j^{\mathrm{gr}} = v_j^{\mathrm{gr}}(r)$ (where j is the mode number) along the path. These quantities are shown in Fig. 2 as the functions of r for mode numbers $j = 1, 15, 25, 35$.

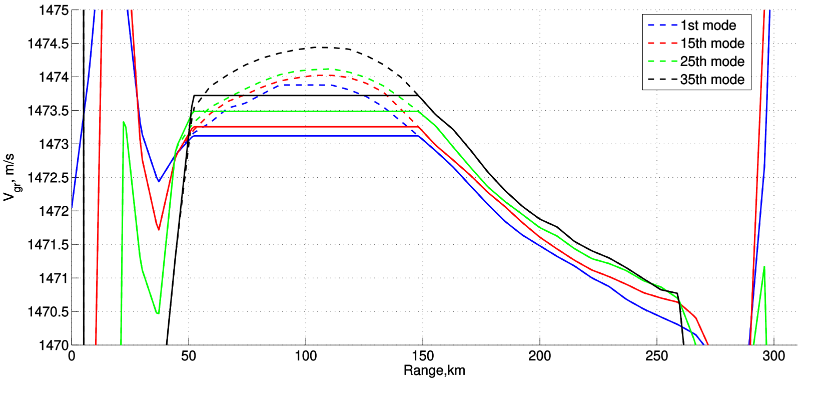


Fig. 2: Modal group velocities $v_j^{\mathrm{gr}} = v_j^{\mathrm{gr}}(r)$ as the functions of range r for mode numbers $j = 1, 15, 25, 35$ without the eddy (solid lines) and in its presence (dashed lines).

It is assumed that effective velocities of propagation of pulses in the absence of the eddy are known, and that the pulses contain the information on transmission time (the clocks at the source and the receiver are synchronized). Under this assumption it is estimated that the presence of the eddy can lead to the inaccuracy of range estimation of about 15 m (for the great circle distance between the source and the receiver of 250 km). The dependence of this error on the depth of the reception point is also considered.

We also estimate the influence of horizontal refraction on the accuracy of ranging problem solution.

The reported study was supported by the Russian Foundation for Basic Research under the contracts № 18-05-00057_a and № 18-35-20081_mol_a_ved.

References

[1] R. N. Baer, Propagation through a three-dimensional eddy including effects on an array, *JASA*, **69**, 70–75 (1981).

[2] F. Jensen, W. Kuperman, M. Porter, H. Schmidt, *Computational Ocean Acoustics*, Springer, New York, et al., 2011.

Propagation of TM waves in a shielded dielectric layer with inhomogeneous cubic nonlinearity

Tikhov S.V., Valovik D.V.
Department of Mathematics and Supercomputing, Penza State University, Penza, 440026, Russia
e-mail: `tik.stanislav2015@yandex.ru, dvalovik@mail.ru`

We consider the propagation of a monochromatic transverse-magnetic (TM) wave in the dielectric layer $\Sigma := \{(x, y, z) : 0 \leqslant x \leqslant h, (y, z) \in \mathbb{R}^2\}$ with perfectly conducted walls [1]. The TM wave has the form $(\mathbf{E}, \mathbf{H})e^{-i\omega t}$, where

$$\mathbf{E} = \big(\mathrm{E}_x(x)e^{i\gamma z}, 0, \mathrm{E}_z(x)e^{i\gamma z}\big)^\top, \quad \mathbf{H} = \big(0, \mathrm{H}_y(x)e^{i\gamma z}, 0\big)^\top \tag{1}$$

are the complex amplitudes; ω is the circular frequency; $(\,\cdot\,)^\top$ is the transposition operation; γ is an unknown (real) spectral parameter (propagation constant of a guided wave); $\mathrm{E}_x, \mathrm{E}_z, \mathrm{H}_y$ are unknown functions [1, 2].

The waveguide Σ is located in the Cartesian coordinates $Oxyz$. At the boundaries $x = 0$, $x = h$, the waveguide has perfectly conductive walls. Inside the waveguide Σ, the permittivity is described by the formula

$$\epsilon = \epsilon_1 + a|\mathbf{E}|^2, \tag{2}$$

where $\epsilon_1 \equiv \epsilon_1(x) \in C^1[0, h]$ is monotonically increasing function, $\epsilon_1(0) = c > 0$, and $a > 0$ is real constant. Everywhere $\mu = \mu_0$, where μ_0 is the magnetic permeability of free space [1, 2].

Complex amplitudes (1) satisfy Maxwell's equations

$$\operatorname{rot}\mathbf{H} = -i\omega\varepsilon\mathbf{E}, \quad \operatorname{rot}\mathbf{E} = i\omega\mu\mathbf{H}; \tag{3}$$

tangential components of the electric field $\mathbf{E}$ vanish on the perfectly conductive walls. We also impose an additional local condition on $\mathrm{E}_x(x)$ at the point $x = 0$, see [3, 4] for details.

For the considered problem, a rigorous analytical approach is suggested for the first time. It is proved that even for small values of the nonlinearity coefficients a, the nonlinear problem has infinitely many nonperturbative solutions (propagation constants and eigenmodes), whereas the corresponding linear problem always has a finite number of solutions. Asymptotic distribution of the propagation constants is found, zeros of the eigenmodes are determined. Similar results for shielded plane waveguide with homogeneous isotropic/anisotropic cubic nonlinearity are represented in [3, 4].

References

[1] P. N. Eleonskii, L. G. Oganes'yants, V. P. Silin, Cylindrical nonlinear waveguides, *Soviet Physics JETP*, 44–47 (1972).

[2] A. D. Boardman, P. Egan, F. Lederer, U. Langbein, D. Mihalache, *Third-Order Nonlinear Electromagnetic TE and TM Guided Waves*, Elsevier, New York, 1991.

[3] D. V. Valovik, S. V. Tikhov, On the existence of infinitely many eigenvalues in a nonlinear waveguiding theory problem, *Computational Mathematics and Mathematical Physics*, **58**(10), 1600–1609 (2018).

[4] D. V. Valovik, S. V. Tikhov, Asymptotic analysis of a nonlinear eigenvalue problem arising in the waveguide theory, *Differential Equations*, **55**(12), 1554–1569 (2019).

Solution of the two-dimensional Dirac equation with a linear potential and a localized initial condition

Tolchennikov A.A.
Ishlinsky Institute for Problems in Mechanics RAS, Moscow, Russia
e-mail: `tolchennikovaa@gmail.com`

We consider the Cauchy problem with localized initial data for a two-dimensional massless Dirac equation with a linear potential $U(x) = x_1$. The solution can be expressed as integral of special function (functions of parabolic cylinder). We simplify this exact solution asymptotically in some domains.

Joint work with S. Yu. Dobrokhotov. The work was supported by the Russian Science Foundation (grant № 16-11-10282).

References

[1] T. Tudorovskiy, K. J. A. Reijnders, M. I. Katsnelson, Chiral tunneling in single-layer and bilayer graphene, *Physica Scripta*, **146**, 014010 (2012).

[2] S. Yu. Dobrokhotov, A. A. Tolchennikov, Solution of the two-dimensional Dirac equation with a linear potential and a localized initial condition, *Russian Journal of Mathematical Physics*, **26**, 139–151 (2019).

Propagation of VLF waves guided by plane density trough in the magnetosphere

Yashina N.F., Zaboronkova T.M.
Department of Nuclear Physics, R. E. Alekseev Technical University of Nizhny Novgorod, 24 Minin St., Nizhny Novgorod 603950, Russia
e-mail: `gemtityash@mail.ru, t.zaboronkova@rambler.ru`

Krafft C.
Laboratoire de Physique des Plasmas, École Polytechnique, 91128 Palaiseau Cedex, France
e-mail: `catherine.krafft@lpp.polytechnique.fr`

It is well-known that VLF wave energy transport in the upper ionosphere can be effective due to whistler waves guided along field-aligned density irregularities, known as density ducts. The ducts in magnetosphere can be generated by powerful high-frequency heating facilities [1]. Propagation of whistler waves in a magnetized plasma containing multiple small-scale irregularities of enhanced electron density was analyzed in [2]. In this report we study the propagation and interaction of VLF waves guided by plane channel with decreased plasma density relatively to the background plasma. The duct parameters are typical of upper ionosphere. The magnetized plasma is described by dielectric tensor with nonzero off-diagonal elements. We consider the case of resonant magnetized plasma in which the diagonal elements of a dielectric tensor have the different signs. This condition in upper ionosphere is satisfied when the frequency of the waves belongs to the interval between the lower hybrid frequency and the electron gyrofrequency.

The dispersion characteristics and structure of eigen modes guided by slab with decreased plasma density are analyzed. It is shown that, under specific condition, a time-harmonic external electromagnetic field may drive the parametric interaction of guided modes. Note that nonlinear interaction of waves guided by density ducts with parameters typical of low ionosphere have been considered in [3]. The purpose of the present work is to study the interaction of proper modes guided by troughs with parameters typical of upper ionosphere. The growth rate and the threshold amplitude of the external field of the parametric instability of modes propagating in the opposite directions are determined. Numerical results will be reported for some practically interesting cases.

Acknowledgments. This work was supported by the Center national de la recherche scientifique.

References

[1] V. L. Frolov, V. O. Rapoport, G. P. Komrakov, et al., Density ducts formed by heating the Earth's ionosphere with high-power HF radio waves, *Sov. J. Exp. and Theor. Phys. Lett.*, **88**, 790–794 (2008).

[2] I. Yu. Zudin, T. M. Zaboronkova, M. E. Cushcin, et al., Whistler waves' propagation in plasmas with systems of small-scale density irregularities: numerical simulation and theory, *J. Geophys. Res.*, **124**, 4739–4760 (2019).

[3] T. M. Zaboronkova, C. Krafft, N. F. Yashina, Nonlinear interaction of whistler waves in a magnetized plasma with density ducts, *Phys. Plasmas.*, **26**, 102104-1–10 (2019).

Surface waves generation in Sommerfeld antenna problem for the Earth radiolocation

V. Zalipaev, V. Vialov, A. Matveensev
Krylov State Research Centre, St. Petersburg, Russia
e-mail: v.zalipaev@metalab.ifmo.ru

The problem of signal transmission on the basis of electromagnetic wave propagation over a long distances had become vital since the beginning of the twentieth century in various areas of scientific activity and engineering. It does still attract attention of many researchers in the field of radio science, mathematical physics, electric engineering of metamaterials and other areas of modern science. The key question has always been how to explain the surprising ability for the radio signals to propagate over a quite long distances within the frame of Maxwell theory of electromagnetism.

In his famous book [1], Arnold Sommerfeld described the analytical solution to the very important canonical problem of Maxwell theory of electromagnetic wave propagation — a radiation of the vertical and horizontal point dipole antennas located over the Earth surface. He studied both cases of the flat and the spherical Earth surfaces. In general for this problem of the flat Earth surface we have got two dielectric half-spaces with different parameters of the dielectric permittivity and permeability, and the dipole antenna is located in the upper half-space close to the interface. In the book [1] the upper half-space is considered to be the air. For the flat surface, he obtained the exact solution in the form of Sommerfeld integrals representing Hankel transform that is well-known in the mathematical theory of wave propagation. He also discussed the generation of the surface waves, the so called Zenneck waves, in the case when the Earth half-space represents a medium of complex dielectric permittivity $\epsilon = \epsilon' + 4\pi i\sigma/\omega$ with very high value of conductivity σ. Here $\epsilon' > 0$ is the real part of the dielectric permittivity, and ω is the frequency. Now it is very well understood that these waves represent the corresponding residue of a pole for the Sommerfeld integrals of the exact solution to the problem. They are very similar to the Rayleigh surface waves in elasticity theory that propagate along a boundary of elastic half-space. If the point source of the electric dipole located quiet close to the interface $z = 0$ between the half-spaces the surface wave is generated with a significant amplitude. If we neglect the absorption in the radial direction it propagates away from the source along the xy plane attenuating in the amplitude as one over the square root of the distance from the source that is opposite to the bulky wave which attenuates as one over distance. It remains localised exponentially in the orthogonal direction with respect to z. Moreover, the speed of attenuation of the surface waves is different for both half-spaces with different medium parameters.

However, in radio science activity a real dipole antenna is not a point source. In practice it is a vibrator antenna that is thin metallic wire of finite length with a gap in the middle with applied voltage. Thus in this work, for the flat interface using Sommerfeld analytical solution of the point source problem, we derive integral equations of the Pocklington type (see for example [2]) for the electric current of the vertical and horizontal vibrator antennas of finite length to describe excitation of the surface waves. It is worth remarking that recently the approach based on the Pocklington type integral equation was successfully applied to studying the electromagnetic localized modes of linear periodic arrays of thin metallic wires (see [3]). In our problem the antenna is located in the upper half-space that is the air. We pay a particular attention to the special cases when the lower half-space is the metallic medium with finite conductivity or a dielectric medium with low conductivity. As far as the electric current of the thin wire vibrator antenna has been computed numerically we evaluate the near field in the form of Hertz vector integral representations of Sommerfeld exact solution. In the far field zone, applying the steepest descend method of the short-wave approximation to these integral representations, for the wave field we obtain asymptotic expansions of the expanding bulky spherical waves in both half-space as well as the poles contributions of excited surface waves. Developing the asymptotic analysis we derive approximate formulas for the surface waves excitation coefficients incorporating the electric current of the vibration antennas. In the numerical analysis we compare the power of the surface wave generated by both types of vibrator antenna for the two different types of the lower medium. We also investigate a question on how close the vibrator antenna could be located with respect to the interface while securing that the long-wave approximation for the Pocklington integral equation is still valid.

References

[1] A. Sommerfeld, *Partial Differential Equations in Physics. Lectures on Theoretical Physics. Volume 6*, Academic Press, 1949.

[2] R. Mittra, *Computer Techniques for Electromagnetics*, Pergamon Press, 1973.

[3] V. Zalipaev, S. Kosulnikov, Guided electromagnetic waves for periodic arrays of thin metallic wires near an interface between two dielectric media, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, **475**(2221), 20180399 (2019).

Asymptotic analysis of plane-wave diffraction by a truncated, sound-hard wedge by means of the method of parabolic wave-equation

Zhu N.Y.

Institute of Radio Frequency Technology, University of Stuttgart, Stuttgart, Germany

e-mail: `zhu@ihf.uni-stuttgart.de`

The scattering of a plane wave at a polygon is a challenging canonical problem in the classical theory of diffraction (see for example [1]). In this work we deal with a truncated, sound-hard wedge, that is to say a polygon with a finite and two semi-infinite sides. We tackle this problem with the aid of the method of parabolic wave-equation, which has been applied to sound diffraction by a finite strip in [2].

As is well known, the total field consists of the ray-optical field and edge-diffracted fields. Each of the edge-diffracted rays is represented as a product of a two factors: a fast-oscillating exponential function describing wave motion in the radial direction of the respective cylindrical co-ordinate system and a slowly-varying complex amplitude. The complex amplitudes meet asymptotically the parabolic wave-equation in their respective ray co-ordinates (here cylindrical ones). As shown for example in [2], the complex amplitudes are related to their spectra in terms of infinite integrals which are the little known Fresnel transforms [3]. Then we derive, from the prescribed conditions at the surface of the truncated wedge, the shadow boundaries of the ray-optics and two auxiliary boundaries, a system of coupled functional equations for the spectra of the complex amplitudes. On use of some properties of the spectra and the relationship between the Fresnel and Fourier transforms, we arrive at a system of two coupled integral equations of the second kind.

It is noted that when the length of the finite side of the wedge increases, the coupling of the two integral equations decreases, leading to two independent algebraic equations in the limiting case. On the other side, when the finite side shrinks to zero, the coupled integral equations reduce to coupled algebraic ones which lend themselves to exact solutions. In both cases we recover the known exact solutions to wave diffraction by the appropriate conventional wedge in the framework of the method of parabolic wave-equation.

In the general case, the unknown spectra are expanded in a infinite series containg Hermite polynomials. By making use of a property of the Fresnel transform [3], the coupled system of two integral equations are converted into two algebraic equations which are solved numerically in an efficient manner.

References

[1] J. M. L. Bernard, A spectral approach for scattering by impedance polygons, *Q. Jl Mech. Appl. Math.*, **50**, 517–550 (2006).

[2] A. V. Popov, Transverse diffusion method in the problem of sound diffraction by a strip, *Sov. Phys. Acoust.*, **19**, 375–378 (1974).

[3] F. Gori, Why is the Fresnel Transform So Little Known? *Current Trends in Optics*, Ed.: J. C. Dainty, 139–148, Academic Press, London, 1994.

Diffraction by a jump of curvature: Wavefield at a moderate distance near the limit ray

Ekaterina A. Zlobina[1], Aleksei P. Kiselev[1,2,3]

[1]St. Petersburg State University, St. Petersburg, Russia

[2]St. Petersburg Department of V. A. Steklov Institute of Mathematics of the Russian Academy of Sciences, St. Petersburg, Russia

[3]Institute for Problems in Mechanical Engineering of the Russian Academy of Sciences, St. Petersburg, Russia

e-mail: `ezlobina2@yandex.ru, aleksei.kiselev@gmail.com`

We are concerned with finding formulas describing high-frequency diffraction by a contour with a jump of curvature. In earlier research [1, 2] we have shown that rigorous boundary-layer approach allows description of the wavefield near the limit ray in terms of the parabolic cylinder function D_{-3}. This result is valid in a small neighborhood of the singular point. Now we extend formulas to a wider area along the limit ray.

The research was supported by RFBR grant № 20-01-00627.

References

[1] E. A. Zlobina, A. P. Kiselev, High frequency diffraction by a contour with a jump of curvature, *Proceedings of the International Conference "Days on Diffraction 2018"*, 325–328 (2018).

[2] E. A. Zlobina, A. P. Kiselev, Boundary-layer approach to high-frequency diffraction by a jump of curvature, *Wave Motion*, **96**, 102571 (2020).

Author index