Sirius University of Science and Technology

Sirius Mathematics Center

International Conference

Complex Approximations, Orthogonal Polynomials and Applications

6–12 June 2021

Programme and Abstracts

Sochi, 2021

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Complex Approximations, Orthogonal Polynomials and Applications
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International Conference "Complex Approximations, Orthogonal Polynomials and Applications": https://caopa.tilda.ws

Speakers :

- Alexander Aptekarev, *Keldysh Institute of Applied Mathematics RAS, Moscow, Russia*
- Walter Van Assche, *Katholieke Universiteit Leuven, Belgium*
- Bernhard Beckermann, *Université de Lille, France*
- Elena Belega, *Moscow State University, Russia*
- Valery Beloshapka, *Moscow State University, Russia*
- Roman Bessonov, *St. Petersburg State University, Russia*
- Sergei Bezrodnykh, *Federal Research Center "Computer Science and Control" RAS, Moscow, Russia*
- Andrei Bogatyrev, *Institute of Numerical Mathematics RAS, Moscow, Russia*
- Jacob Christiansen, *Lund University, Sweden*
- Alfredo Deaño, *Universidad Carlos III de Madrid, Spain*
- Sergey Denisov, *University of Wisconsin–Madison, USA*
- Peter Dragnev, *Purdue University Fort Wayne, Indiana, USA*
- Alexander Dyachenko, *Universität Konstanz, Germany*
- Konstantin Fedorovskiy, *Moscow State University, Russia*
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- Arno Kuijlaars, *Katholieke Universiteit Leuven, Belgium*
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- Francisco Marcellán, *Universidad Carlos III de Madrid, Spain*
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- Béla Nagy, *University of Szeged, Hungary*
- Semen Nasyrov, *Kazan Federal University, Russia*
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- Armen Sergeev, *Steklov Mathematical Institute RAS, Moscow, Russia*
- Vladimir Sorokin, *Moscow State University, Russia*
- Maria Stepanova, *Moscow State University, Russia*
- Nikos Stylianopoulos, *University of Cyprus*
- Vilmos Totik, *University of Szeged, Hungary*
- August Tsikh, *Siberian Federal University, Krasnoyarsk, Russia*
- Anna Tsvetkova, *Institute for Problems in Mechanics RAS, Moscow, Russia*
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- Franck Wielonsky, *Université Aix-Marseille, France*
- Maxim Yattselev, *Indiana University–Purdue University Indianapolis, USA*
- Peter Yuditskii, *Johannes Kepler University, Linz, Austria*

CONFERENCE PROGRAMME

MONDAY 7 JUNE

09^{00} — 09^{30} REGISTRATION

09^{30} — 10^{10} Roman Bessonov (St. Petersburg State University, Russia). *An almost periodic model for general reflectionless spectral data.*

10^{15} — 10^{40} Sergei Kalmykov (Shanghai Jiao Tong University, China). *On the uniform convergence of Green's functions.*

BREAK

11^{00} — 11^{15} OPENING CEREMONY

11^{15} — 11^{40} Vilmos Totik (University of Szeged, Hungary). *Multiplicity of zeros of polynomials.*

11^{45} — 12^{25} Peter Yuditskii (Johannes Kepler University, Linz, Austria). *Pointwise Remez inequality.*

12^{30} — 12^{55} Anna Kononova (St. Petersburg State University, Russia). *Convergence rate for weighted polynomial approximation on the real line.*

LUNCH

14^{30} — 14^{55} Bernhard Beckermann and Ana Matos (Université de Lille, France). *Solving the signed equilibrium problem on several real intervals.*

15^{00} — 15^{25} Eugene Tyrtyshnikov (Institute of Numerical Mathematics, Moscow, Russia). *Tensors, polynomials and rank increment conjecture.*

15^{30} — 15^{55} Andrei Bogatyrev (Institute of Numerical Mathematics, Moscow, Russia). *Stability polynomials for explicit Runge–Kutta methods: optimal and damped.*

BREAK

16^{30} — 16^{55} Maxim Yattselev (Indiana University–Purdue University Indianapolis, USA). *On $L^2_{\mathbb{R}}$-best rational approximants to Markov functions on several intervals.*

17^{00} — 17^{25} Sergey Denisov (University of Wisconsin–Madison, USA). *Subharmonic functions in spectral theory and multidimensional scattering.*

17^{30} — 17^{55} Galina Filipuk (University of Warsaw, Poland). *Aspects of nonlinear differential equations.*

TUESDAY 8 JUNE

09^{30} — 10^{10} Sergei Bezrodnykh (Federal Research Center "Computer Science and Control", Moscow, Russia). *Analytic continuation of the multiple hypergeometric functions.*

10^{15} — 10^{40} Semen Nasyrov (Kazan Federal University, Russia). *Properties of level curves of a function generated by an Abel integral on three-sheeted torus.*

BREAK

11^{00} — 11^{40} Arno Kuijlaars (Katholieke Universiteit Leuven, Belgium). *Matrix-valued orthogonal polynomials related to hexagon tilings.*

11^{45} — 12^{25} Alfredo Deaño (Universidad Carlos III de Madrid, Spain). *Global phase portrait and large degree asymptotics for the kissing polynomials.*

12^{30} — 12^{55} Wayne Lawton (Siberian Federal University, Krasnoyarsk, Russia). *Approximations and almost periodicity.*

LUNCH

14^{30} — 14^{55} Francisco Marcellán (Universidad Carlos III de Madrid, Spain). *From OPRL to OPUC. A matrix approach beyond the Szegő transformations.*

15^{00} — 15^{25} Mikhail Tyaglov (Shanghai Jiao Tong University, China). *Polynomial solutions of linear differential operators and Bochner's theorem.*

15^{30} — 15^{55} Franck Wielonsky (Université Aix-Marseille, France). *Riesz energy problems in unbounded sets of* $\mathbb{R}^d$.

BREAK

16^{30} — 16^{55} Miguel Piñar (Universidad de Granada, Spain). *Jackson-type estimates on the weighted unit ball.*

17^{00} — 17^{25} Jacob Christiansen (Lund University, Sweden). *Chebyshev polynomials in the complex plane.*

17^{30} — 17^{55} Peter Dragnev (Purdue University Fort Wayne, Indiana, USA). *On the best uniform polynomial approximation to the checkmark function.*

18^{30} WELCOME PARTY

WEDNESDAY 9 JUNE

09^{30} — 10^{10} Anna Tsvetkova (Institute for Problems in Mechanics, Moscow, Russia). *Real-valued semiclassical approximation for the asymptotics with complex-valued phases and its application to multiple orthogonal Hermite polynomials.*

10^{15} — 10^{40} Rostyslav Kozhan (Uppsala University, Sweden). *Differential equation for the limit of the nearest neighbour recursion coefficients for MORPL.*

BREAK

11^{00} — 11^{40} Walter Van Assche (Katholieke Universiteit Leuven, Belgium). *Hermite–Padé approximation and the number π.*

11^{45} — 12^{10} Valery Kalyagin (Higher School of Economics, N. Novgorod, Russia). *On some results in multiple orthogonal polynomials and its applications.*

LUNCH

13^{15} TRIP TO KRASNAYA POLYANA

THURSDAY 10 JUNE

09^{30} — 10^{10} Armen Sergeev (Steklov Mathematical Institute, Moscow, Russia). *From Ginzburg–Landau vortices to Seiberg–Witten equations.*

10^{15} — 10^{40} Teresa Pérez (Universidad de Granada, Spain). *Multivariate mixed orthogonal functions satisfying three term relations.*

BREAK

11^{00} — 11^{25} Guillermo Lopez Lagomasino (Universidad Carlos III de Madrid, Spain). *On Markov's theorem for multi-level Hermite–Padé approximants of a Nikishin system.*

11^{30} — 11^{55} Luis González Ricardo (Universidad Carlos III de Madrid, Spain). *Ratio asymptotic of generalized multi-level Hermite–Padé polynomials.*

12^{00} — 12^{25} Vladimir Lysov (Keldysh Institute of Applied Mathematics, Moscow, Russia). *On algebraic properties of the mixed type approximants.*

12^{30} — 12^{55} Elijah Lopatin (Steklov Mathematical Institute, Moscow, Russia). *On the generalisation of the scalar approach to the weak asymptotics of Hermite–Padé polynomials: some recent achievements.*

LUNCH

14^{30} — 14^{55} Ana Loureiro (University of Kent, Canterbury, UK). *Ratio asymptotics for symmetric multiple orthogonal polynomials.*

15^{00} — 15^{25} Alexander Dyachenko (Universität Konstanz, Germany). *Discrete multiple orthogonal polynomials on shifted lattices.*

15^{30} — 15^{55} Maria Lapik (Keldysh Institute of Applied Mathematics, Moscow, Russia). *On the asymptotics of orthogonal measure for special polynomials in the problem of radiation scattering.*

BREAK

SEVERAL VARIABLES SECTION

16^{30} — 16^{55} Valery Beloshapka (Moscow State University, Russia). *Birationality of the automorphisms of the model surface.*

17^{00} — 17^{25} Maria Stepanova (Moscow State University, Russia). *On functions of finite analytical complexity.*

17^{30} — 19^{00} Discussion and open problems by August Tsikh, Valery Beloshapka, Maria Stepanova, Alexander Mkrtchyan, Dmitriy Pochekutov, Ekaterina Kleshkova, Sergey Feklistov et al.

HYDRODYNAMICS SECTION

16^{30} — 16^{55} Pavel Lushnikov (University of New Mexico, Albuquerque, USA). *Dynamics of complex singularities in fluid dynamics and analytical continuation by rational approximants.*

17^{00} — 17^{25} Pavel Plotnikov (Lavrentyev Institute of Hydrodynamics, Novosibirsk, Russia). *Stokes waves of extreme form.*

17^{30} — 19^{00} Discussion and open problems by Andrei Afendikov, Pavel Lushnikov, Pavel Plotnikov, Alexander Aptekarev, Elena Belega, Alexei Ilyin, Tatiana Dudnikova et al.

FRIDAY 11 JUNE

09^{30} — 10^{10} Andrei Martinez-Finkelshtein (Baylor University, Waco, Texas, USA). *Poncelet–Darboux, Kippenhahn, and Szegő: projective geometry, matrices and orthogonal polynomials.*

10^{15} — 10^{40} Béla Nagy (University of Szeged, Hungary). *Bernstein type inequalities and potential theory.*

BREAK

11^{00} — 11^{25} Sergei Khrushchev (Satbayev University, Almaty, Kazakhstan). *Uniformly convergent Fourier series with universal power parts on closed subsets of measure zero.*

11^{30} — 11^{55} Alexei Ilyin (Keldysh Institute of Applied Mathematics, Moscow, Russia). *Optimal bounds for the attractor dimension for 3D damped regularized Euler equations.*

12^{00} — 12^{25} Konstantin Fedorovskiy (Moscow State University, Russia). *Chui's conjecture in Bergman spaces.*

12^{30} — 13^{10} Nikos Stylianopoulos (University of Cyprus, Nicosia). *Bergman, Szegő, Faber and Christoffel: boundary behavior.*

13^{15} CLOSING CEREMONY

Abstracts of the talks

Hermite–Padé approximation and the number π

Walter Van Assche

Katholieke Universiteit Leuven

It is well known that π is an irrational number and that it is also a transcendental number, but there are still many open problems for this interesting number. It is still not known how well it can be approximated by rational numbers. Last year Zeilberger and Zudilin (2020) found the best upper bound so far: the measure of irrationality of π is bounded from above by 7.103205334137. They improved an earlier upper bound of Salikhov from 2008, and before him the best upper bound was obtained by Hata (1993). These upper bounds were obtained by analyzing certain integrals of rational functions over contours in the complex plane.

In my talk I will show that these integrals are closely related to an Hermite–Padé approximation problem for a pair of Markov functions. We will investigate this Hermite–Padé approximation in some detail using the corresponding vector equilibrium problem, algebraic functions satisfying a cubic equation and we describe the Riemann–Hilbert problem for this Hermite–Padé approximation problem.

Solving the signed equilibrium problem on several real intervals

Bernhard Beckermann and Ana Matos

Université de Lille

Given the union I of real disjoint intervals and a smooth external field Q on I, the signed equilibrium problem consist of finding a signed measure μ of mass 1 supported on I such that $U^\mu + Q$ is equal to a constant on I. Taking derivatives, we may reformulate this problem as an integral equation with a Cauchy kernel, and can solve the problem via a block polynomial spectral method via Chebyshev polynomials. However, small gaps between consecutive subintervals deteriorate the rate of convergence. Instead, we suggest a new spectral method based on Chebyshev rational orthogonal functions, leading to much smaller systems of linear equations. We present several numerical examples, and combine this technique with the iterated balayage algorithm of Dragnev, in order to solve also positive equilibrium problems.

The Shannon entropy of high-dimensional harmonic systems

Elena Belega

Moscow State University

In this talk we discuss some properties of the Shannon entropy of the discrete stationary states of the high-dimensional harmonic (oscillator-like) systems. Our approach is based on the asymptotics of some entropy-like integral functionals of the orthogonal polynomials when the polynomial parameter answering for the system's dimensionality is large. In particular we show that the Shannon entropy of the D-dimensional harmonic systems has a logarithmic growth rate of the type $D \log D$ when $D \to \infty$.

The talk is based on a joint work with A. I. Aptekarev, J. S. Dehesa and I. V. Toranzo.

Birationality of the automorphisms of the model surface

Valery Beloshapka

Moscow State University

We will discuss that the group of holomorphic automorphisms of holomorphically homogeneous nondegenerate (finite Bloom–Graham type + holomorphic nondegenaracy) model surface is a subgroup of the group of birational automorphisms of the ambient space (Cremona group) with uniformly bounded degree. We will give an estimate of the degree of the automorphisms in terms of the dimension of the ambient space.

An almost periodic model for general reflectionless spectral data

Roman Bessonov

St. Petersburg State University

A basic feature of second order almost periodic differential and finite-difference operators is the reflectionless property of their Weyl functions. Conversely, each regular enough pair of reflectionless Nevanlinna functions generate an almost periodic operator on the real line whose half-line Weyl functions coincide with the given pair. Until recently, for operators with unbounded spectra, this scheme worked under quite restrictive assumptions on the "quality" of the spectrum. We extend it to cover all homogeneous spectra, and, more generally, to all spectra satisfying Widom and DCT conditions.

This is a joint work with M. Lukic and P. Yuditskii. The author is partially supported by the Russian Science Foundation grant 19-11-0005.

Analytic continuation of the multiple hypergeometric functions

Sergei Bezrodnykh

Federal Research Center "Computer Science and Control"

A wide class of hypergeometric functions in several variables $\mathbf{z} = (z_1, z_2, \dots, z_N) \in \mathbb{C}^N$ is defined with the help of the Horn series [1–3], which has the form:

$$\Phi^{(N)}(\mathbf{z}) = \sum\nolimits_{\mathbf{k}\in\mathbb{Z}^N} \Lambda(\mathbf{k})\mathbf{z}^{\mathbf{k}}; \tag{1}$$

here $\mathbf{k} = (k_1, k_2, \dots, k_N)$ is the multi-indices, $\mathbf{z}^{\mathbf{k}} := z_1^{k_1} z_2^{k_2} \cdots z_N^{k_N}$, and the coefficients $\Lambda(\mathbf{k})$ are such that the ratio of any two adjacent is a rational function of the components of the summation index. In other words, for all $j = \overline{1, N}$ the relations are fulfilled: $\Lambda(\mathbf{k} + \mathbf{e}_j)/\Lambda(\mathbf{k}) = P_j(\mathbf{k})/Q_j(\mathbf{k})$, $j = \overline{1, N}$, where P_j and Q_j are some polynomials in the N variables $k_1, k_2, \dots, k_N$ and $\mathbf{e}_j = (0, \dots, 1, \dots, 0)$ denote the vectors with jth component equal to 1.

The talk describes the approach proposed in [4] for deriving formulas for the analytic continuation of series (1) with respect to the variables $\mathbf{z}$ into the entire complex space $\mathbb{C}^N$ in the form of linear combinations $\Phi^{(N)}(\mathbf{z}) = \sum_m A_m u_m(\mathbf{z})$, where $u_m(\mathbf{z})$ are hypergeometric series of the Horn type satisfying the same system of partial differential equations as the series (1) and A_m are some coefficients. The implementation of this approach is demonstrated by the example of the Lauricella hypergeometric function $F_D^{(N)}$. In the unit polydisk $\mathbb{U}^N := \{|z_j| < 1,\ j = \overline{1, N}\}$, this function is defined by the following series, see [5], [6]:

$$F_D^{(N)}(\mathbf{a}; b,\ c;\ \mathbf{z}) := \sum_{|\mathbf{k}|=0}^{\infty} \frac{(b)_{|\mathbf{k}|}(a_1)_{k_1} \cdots (a_N)_{k_N}}{(c)_{|\mathbf{k}|} k_1! \cdots k_N!}\, \mathbf{z}^{\mathbf{k}}; \tag{2}$$

here the complex values $(a_1, \dots, a_N) =: \mathbf{a}$, b, and c play the role of parameters, $c \notin \mathbb{Z}^-$, $|\mathbf{k}| := \sum_{j=1}^N k_j$, and $k_j \geq 0$, $j = \overline{1, N}$, the Pochhammer symbol is defined as $(a)_m := \Gamma(a + m)/\Gamma(a) = a(a + 1) \cdots (a + m - 1)$.

In [7], we have constructed a complete set of formulas for the analytic continuation of series (1) for an arbitrary N into the exterior of the unit polydisk. Such formulas represent the function $F_D^{(N)}$ in suitable subdomains of $\mathbb{C}^N$ as linear combinations of hypergeometric series that are solutions of the following system [5], [6] of partial differential equations:

$$z_j(1 - z_j)\frac{\partial^2 u}{\partial {z_j}^2} + (1 - z_j)\sum\nolimits_{k=1}^{\prime N} z_k \frac{\partial^2 u}{\partial z_j \partial z_k} +$$
$$+\Big[c - (1 + a_j + b)z_j\Big]\frac{\partial u}{\partial z_j} - a_j \sum\nolimits_{k=1}^{\prime N} z_k \frac{\partial u}{\partial z_k} - a_j b\, u = 0, \qquad j = \overline{1, N},$$

which the function $F_D^{(N)}$ satisfies; here a prime on a summation sign means that the sum is taken for $k \neq j$. The convergence domains of the found continuation formulas together cover $\mathbb{C}^N \setminus \mathbb{U}^N$.

We give an application of the obtained results on the analytic continuation of the Lauricella function $F_D^{(N)}$ to effective computation of conformal map of polygonal domains in the crowding situation.

The work is financially supported by RFBR, proj. 19-07-00750.

References
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[2] Gel'fand I. M., Graev M. I., Retakh V. S., "General hypergeometric systems of equations and series of hypergeometric type." *Russ. Math. Surv.,* 47(4): 1–88, 1992.
[3] Sadykov T. M., Tsikh A. K., *Hypergeometric and algebraic functions in several variables.* Nauka, Moscow, 2014. (In Russian).
[4] Bezrodnykh S. I., "Analytic continuation of the Horn hypergeometric series with an arbitrary number of variables." *Int. Transf. Spec. Funct.,* 31(10): 788–803, 2020.
[5] Lauricella G., "Sulle funzioni ipergeometriche a più variabili." *Rend. Circ. Mat. Palermo,* 7: 111–158, 1893.
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Stability polynomials for explicit Runge–Kutta methods: optimal and damped

Andrei Bogatyrev

Institute of Numerical Mathematics

Explicit methods for numerical integration of ordinary differential equations are very appealing from the viewpoint of computations: they are cost efficient and ideally suit for massive parallel computers. As usual, one has to pay for the advantages: stability properties of explicit methods are rather poor and the Courant time steps which guarantee the stability may become terribly small especially for the ODEs obtained from the spatial discretization of PDEs. Possible solution is to make variable time steps which brings us to a series of nonclassical optimization problems for polynomials.

Chebyshev polynomials in the complex plane

Jacob Christiansen

Lund University

A classical problem that goes back to Chebyshev is to approximate x^n by polynomials of lower degree on some compact interval. The monic polynomials T_n of least deviation from zero on some infinite compact set $\mathsf{E} \subset \mathbb{C}$ hence bear the name of Chebyshev. In the talk, I will discuss results about the zeros of T_n and their asymptotic behavior when E is connected. I will also discuss bounds on the norms $\|T_n\|_{\mathsf{E}}$ and present some open problems.

Global phase portrait and large degree asymptotics for the kissing polynomials

Alfredo Deaño

Universidad Carlos III de Madrid

In this talk we present recent results on orthogonal polynomials (OPs) with respect to the weight function $w(z;s) = e^{-sz}$ on $[-1,1]$, where $s \in \mathbb{C}$ is an arbitrary complex parameter. We are particularly interested in the limit zero distribution and asymptotic behavior of recurrence coefficients as $n \to \infty$. To investigate this, we determine the geometry of breaking points and breaking curves in the s plane, which separate regions where different asymptotic behaviors occur.

References

[1] Barhoumi A., Celsus A. F., Deaño A., "Global phase portrait and large degree asymptotics for the kissing polynomials." arXiv:2008.08724 *to appear in Stud. Appl. Math.*

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Subharmonic functions in spectral theory and multidimensional scattering

Sergey Denisov

University of Wisconsin–Madison

We will discuss how the elementary theory of subharmonic functions and estimates on harmonic measure can be used to study one-dimensional Schrödinger operators with operator-valued potential and scattering theory for multidimensional problems.

On the best uniform polynomial approximation to the checkmark function

Peter Dragnev

Purdue University Fort Wayne

In this talk we shall consider the best uniform polynomial approximation of the checkmark function $f(x) = |x-\alpha|$ as α varies in $(-1, 1)$. For each fixed degree n, the minimax error $E_n(\alpha)$ is shown to be piecewise analytic in α and the best approximants stem from classical polynomials of Chebyshev, Zolotarev, Krein, etc. In addition, $E_n(\alpha)$ is shown to feature $n-1$ piecewise linear decreasing/increasing sections, called V-shapes. The points of the alternation set are proven to be monotone increasing in α and their dynamics are completely characterized. We also prove a conjecture of Shekhtman that for odd n, $E_n(\alpha)$ has a local maximum at $\alpha = 0$.

This is a joint work with Alan Legg (PUFW) and Ramon Orive (ULL), arXiv:2102.09502.

Discrete multiple orthogonal polynomials on shifted lattices

Alexander Dyachenko

University of Konstanz

There are many ways to define multiple orthogonal polynomials with respect to the classical continuous weights. Bearing in mind a deep connection between the classical discrete and continuous orthogonality, we adapt to the discrete case the approach as in [1-3] preserving a kind of the Rodrigues formula. Our work [4] introduced a new class of polynomials of multiple orthogonality with respect to the product of classical discrete weights on integer lattices with noninteger shifts.

This talk is devoted to further progress in this direction for the case of two measures. In particular, we obtain coefficients for the four-term recurrence relations connecting polynomials with indices on “diagonals” (including the “step line”). The initial conditions for these relations are presented by semi-classical extensions of discrete orthogonal polynomials studied in [5-7].

This is a joint work with Vladimir Lysov.

References

[1] Aptekarev A. I., “Multiple orthogonal polynomials.” *J. Comput. Appl. Math.,* 99 (1–2): 423–447, 1998.

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Chui's conjecture in Bergman spaces

Konstantin Fedorovskiy

Moscow State University

In the talk we present a solution of Chui's problem on the simplest fractions (i.e., sums of Cauchy kernels with unit coefficients) in weighted (Hilbert) Bergman spaces. Namely, for a wide class of weights, it will be shown that for every N, the simplest fractions with N poles on the unit circle have minimal norm if and only if the poles are equidistributed on the circle. We present sharp asymptotics of these norms. Finally, we describe the closure of the simplest fractions in weighted Bergman spaces.

The talk is based on a joint work with E. Abakumov (Univ. Gustave Eiffel) and A. Borichev (Aix–Marseille Univ.).

Aspects of nonlinear differential equations

Galina Filipuk

University of Warsaw

Painlevé equations are nonlinear second order differential equations solutions of which have no movable critical points. They have a lot of nice properties. In this talk I shall mainly review connection between solutions of the Painlevé equations and recurrence coefficients of semi-classical orthogonal polynomials.

Ratio asymptotic of generalized multi-level Hermite–Padé polynomials

Luis González Ricardo

Universidad Carlos III de Madrid

In this talk we present recent results on the ratio asymptotic of a family of multi-orthogonal polynomials associated to a mixed-type Hermite–Padé approximation scheme for Nikishin systems, also known as multi-level Hermite–Padé approximation. These results expand those obtained in [1] and are related to the ones obtained in [2].

This is a joint work with G. López Lagomasino and S. Medina Peralta.

References

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Optimal bounds for the attractor dimension for 3D damped regularized Euler equations

Alexei Ilyin

Keldysh Institute of Applied Mathematics and Sirius Mathematical Center

The dependence of the fractal dimension of global attractors for the regularized damped 3D Euler equations on the regularization parameter $\alpha > 0$ is studied. We present explicit upper bounds for this dimension for the case of the whole space, periodic boundary conditions, and the case of a bounded domain with Dirichlet boundary conditions. The sharpness of these estimates when $\alpha \to 0$ (which corresponds in the limit to the classical Euler equations) is demonstrated on the 3D Kolmogorov flows on a torus.

On the uniform convergence of Green's functions

Sergei Kalmykov

Shanghai Jiao Tong University

In this talk, we discuss the uniform convergence of Green's functions of regular planar domains converging in the sense of kernel, provided the limit domain is also regular, and the connectivity is uniformly bounded. Also, we consider several examples when uniform convergence fails (on the complex sphere and in $\mathbb{R}^n$).

The talk is based on joint work with L. V. Kovalev.

On some results in multiple orthogonal polynomials and its applications

Valeriy Kalyagin

National Research University Higher School of Economics

In this talk I will give a short overview on some results in multiple orthogonal polynomials and its applications. In particular, the following topics will be partially covered: strong asymptotics of multiple orthogonal polynomials for Angelesko and Nikishin systems, higher-order recurrences and asymptotics of multiple orthogonal polynomials, Padé approximants for functions with branch points and strong asymptotics of Nuttall–Stahl polynomials, large limit of random matrices and multiple orthogonal polynomials.

Uniformly convergent Fourier series with universal power parts on closed subsets of measure zero

Sergei Khrushchev

Satbayev University

Given a closed subset E of Lebesgue measure zero on the unit circle $\mathbb{T}$ there is a function f on $\mathbb{T}$ with uniformly convergent symmetric Fourier series

$$S_n(f,\zeta) = \sum_{k=-n}^{n} \hat{f}(k)\zeta^k \underset{\mathbb{T}}{\rightrightarrows} f(\zeta),$$

such that for every continuous function g on E, there is a subsequence of partial power sums

$$S_n^+(f,\zeta) = \sum_{k=0}^{n} \hat{f}(k)\zeta^k$$

of f, which converges to g uniformly on E. Here

$$\hat{f}(k) = \int_{\mathbb{T}} \bar{\zeta}^k f(\zeta)\, dm(\zeta),$$

and m is the normalized Lebesgue measure on $\mathbb{T}$.

Convergence rate for weighted polynomial approximation on the real line

Anna Kononova

St. Petersburg State University

In this talk we discuss a quantitative version of Bernstein's approximation problem when the polynomials are dense in weighted spaces on the real line completing a result of S. N. Mergelyan (1960). The talk is based on arXiv:2012.0385.

Differential equation for the limit of the nearest neighbour recursion coefficients for MORPL

Rostyslav Kozhan

Uppsala University

A limiting property of the nearest-neighbor recurrence coefficients for multiple orthogonal polynomials from a Nevai class is investigated. Namely, assuming that the nearest-neighbor coefficients have a limit along rays of the lattice, we describe it in terms of the solution of a system of differential equations.

This is a joint work with A. I. Aptekarev.

Matrix-valued orthogonal polynomials related to hexagon tilings

Arno Kuijlaars

Katholieke Universiteit Leuven

I will discuss a class of matrix-valued orthogonal polynomials (MVOPs) that are related to 2-periodic lozenge tilings of a hexagon. The general model depends on many parameters. In the cases of constant and 2-periodic parameter values we show that the MVOP can be expressed in terms of scalar polynomials with non-Hermitian orthogonality on a closed contour in the complex plane. The 2-periodic hexagon tiling model with a constant parameter has a phase transition in the large size limit. This is reflected in the asymptotic behavior of the MVOP as the degree tends to infinity. The connection with the scalar orthogonal polynomials allows us to find the limiting behavior of the zeros of the determinant of the MVOP. The zeros tend to a curve in the complex plane that has a self-intersection. The zeros of the individual entries of the MVOP show a different behavior and we find the limiting zero distribution of the upper right entry under a geometric condition that we were unable to prove, but that is convincingly supported by numerical evidence.

The talk is based on the preprint arXiv:2104.14822, that is joint work with Alan Groot.

On the asymptotics of orthogonal measure for special polynomials in the problem of radiation scattering

Maria Lapik

Keldysh Institute of Applied Mathematics

For quantum optics models, polynomial Hamiltonians with respect to creation and annihilation operators are used:

$$\hat{H} = \sum_{k=1}^{p} \omega_k a_k^+ a_k^- + \sum_{(\mathbf{r},\mathbf{s}) \in J \subset \mathbb{Z}_+^p} b_{\mathbf{r}\,\mathbf{s}} a^{+\mathbf{r}} a^{-\mathbf{s}} + h.c.,$$

where $a^{\mathbf{r}} = a_1^{r_1} ... a_p^{r_p}$, ω_i and $b_{\mathbf{rs}} = b_{\mathbf{sr}}^*$ are some constants and $h.c.$ are Hermitian conjugated terms of H. The number of different types of particles is p. Standard basis consists of eigenvectors of $\hat{H}_0 = \sum_{k=1}^{p} \omega_k a_k^+ a_k^-$, it is $\{|n\rangle = |n_1\rangle_1 ... |n_p\rangle_p\}$, where $\hat{n}_k = a_k^+ a_k^-$ is k-type particle number operator.

Our goal is to describe the asymptotics of the eigenvalues of the Hamiltonian for large eigenvalues of particle number operator. The system of special non-classical polynomials is introduced for the Hamiltonian diagonalization problem. An asymptotics for orthogonal measure is obtained for these systems of polynomials by logarithmic potential methods when particle number tends to infinity. The exact case of quadratic Hamiltonian will be considered as an explicit example.

This is a joint work with A. I. Aptekarev and Yu. N. Orlov.

Approximations and almost periodicity

Wayne Lawton

Siberian Federal University

Bohr proved that a uniformly almost periodic function $f : \mathbb{R} \to \mathbb{C}$ extends to a continuous function $\widetilde{f} : G(f) \to \mathbb{C}$, where $G(f)$ is a compactification of $\mathbb{R}$ whose Pontryagin dual $\widehat{G(f)}$ is generated by the spectrum $\Omega(f)$. He showed that $\Omega(f)$ is bounded iff f extends to an entire function F of exponential type $\tau(F) < \infty$, and Krein proved that $f \geq 0$ implies $f = |s|^2$ where s extends to an entire function S of exponential type $\tau(S) = \tau(F)/2$ having no zeros in the open upper half plane. The spectral factor s is unique up to a multiplicative factor having modulus 1. Krein and Levin used properties of zero sets to construct f so s is not uniformly almost periodic. We use approximation methods to

(i) characterize f so s is uniformly almost periodic,

(ii) relate s to Helson–Lowdenslager's spectral factorization of $\widetilde{f}$,

(iii) construct uniformly almost periodic h with unbounded Hilbert transform, bounded spectrum, and rank $\widehat{G(h)} = 2$, and

(iv) construct a Bohr–type compactification for and homotopy classification of a class of uniformly periodic discrete subsets of $\mathbb{R}^d$.

On the generalisation of the scalar approach to the weak asymptotics of Hermite–Padé polynomials: some recent achievements

Elijah Lopatin

Steklov Mathematical Institute

Till this moment, the behaviour of the weak asymptotics of zeros of Hermite–Padé polynomials for the Nikishin system (and some more extensive systems of Markov functions) has been usually investigated in the framework of the vector potential equilibrium problem. In 2017 S. Suetin suggested the new approach to this question dealing with the scalar potential problem with external harmonic field stated on a Riemann surface of genus zero (the Riemann sphere); in 2019 it was generalised for the broader class of functions. This generalisation leads to consideration of the scalar potential problem on a Riemann surface of positive genus with respect to the new type of kernels introduced by E. Chirka in 2018. In my talk I will discuss some recent achievements on the implementation of the scalar approach.

On Markov's theorem for multi-level Hermite–Padé approximants of a Nikishin system

Guillermo Lopez Lagomasino

Universidad Carlos III de Madrid

We consider a mixed type Hermite–Padé approximation problem of a Nikishin system of functions in which the generating measures may have unbounded or touching supports and give a Markov type theorem on the convergence of the approximating rational functions.

This is a joint work with Luis Giraldo Gonzalez Ricardo and Sergio Medina Peralta.

Ratio asymptotics for symmetric multiple orthogonal polynomials

Ana Loureiro

University of Kent

At the centre of this talk are polynomials on a single variable satisfying higher order recurrence relations with all recurrence coefficients, except the last one, equal to zero. The polynomials at issue are orthogonal with respect to a vector of measures, are rotational invariant and all the zeros lie on a star in the complex plane. The discussion will focus on their ratio asymptotic behaviour as well as the zero limit distribution.

This is a joint work with Walter Van Assche.

Dynamics of complex singularities in fluid dynamics and analytical continuation by rational approximants

Pavel Lushnikov

University of New Mexico and Landau Institute for Theoretical Physics

A motion of ideal incompressible fluid with a free surface is considered in two-dimensional geometry. A time-dependent conformal mapping of the lower complex half-plane of the auxiliary complex variable w into the area filled with fluid is performed with the real line of w mapped into the free fluid's surface. The fluid dynamics is fully characterized by the motion of the complex singularities in w plane obtained by the analytical continuation into area outside of fluid. We consider both the exact dynamics and the short branch cut approximation of the dynamics with the small parameter being the ratio of the length of the branch cut to the distance between its center and the real line of w. The exact dynamics is shown to conserve the infinite number of integrals of motion which are residues of poles in auxiliary complex variables as well as it results in the motion of power law branch points. Fluid dynamics in short branch cut approximation is reduced to the complex Hopf equation for the complex velocity coupled with the complex transport equation for the conformal mapping. These equations are fully integrable by characteristics producing the infinite family of solutions, including the pairs of moving square root branch points. The solutions are compared with the simulations of the full Eulerian dynamics giving excellent agreement. The numerical continuation is performed into the complex plane using the rational approximants. It allows to compare the dynamics of singularities with the analytical predictions including the existence of multiple integrals of motion. We also analyze the dynamics of singularities and finite time blow up of Constantin–Lax–Majda equation which corresponds to non-potential effective motion of non-viscous fluid with competing convection and vorticity stretching terms. A family of exact solutions is found together with the different types of complex singularities approaching the real line in finite times. These singularities are also recovered by numerical analytical continuation.

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On algebraic properties of the mixed type approximants

Vladimir Lysov

Keldysh Institute of Applied Mathematics

We discuss the mixed type approximants for the Nikishin system of size d and for the entire table of multi-indexes. We show that the approximants satisfy recurrence relations "to the nearest neighbor" of order $d+1$. We also pose a dual interpolation problem and demonstrate the connection of the mixed problems with the type I and II Hermite–Padé approximations.

From OPRL to OPUC. A matrix approach beyond the Szegő transformations

Francisco Marcellán

Universidad Carlos III de Madrid

M. Derevyagin, L. Vinet and A. Zhedanov introduced in [1] a new connection between orthogonal polynomials on the unit circle (OPUC) and orthogonal polynomials on the real line(OPRL). It maps any real CMV matrix into a Jacobi one depending on a real parameter λ. In such a contribution the authors prove that this map yields a natural link between the Jacobi polynomials on the unit circle and the little and big -1 Jacobi polynomials on the real line. They also provide explicit expressions for the measure and orthogonal polynomials associated with the Jacobi matrix in terms of those related to the CMV matrix, but only for the value $\lambda = 1$ which simplifies the connection –*basic DVZ connection*–. However, similar explicit expressions for an arbitrary value of λ –*(general) DVZ connection*– are missing in the above mentioned paper. In this problem we will focus our presentation.

First of all, we will summarize the state of the art, with a special emphasis in the so called Szegő transformations between measures supported on the interval $[-1, 1]$ and some measures supported on the unit circle. Next, we will discuss a new approach to the DVZ connection which formulates it as a two-dimensional eigenproblem by using known properties of CMV matrices. This allows us to go further than the DVZ approach providing explicit relations between the measures and orthogonal polynomials for the general DVZ connection. It turns out that this connection maps a measure on the unit circle into a rational perturbation of an even measure supported on two symmetric intervals of the real line, which reduce to a single interval for the basic DVZ connection, while the perturbation becomes a degree one polynomial. Some instances of the DVZ connection are shown to give new one-parameter families of orthogonal polynomials on the real line. We will follow the approach presented in our recent paper [2].

This is a joint work with M. J. Cantero, L. Moral. L. Velázquez (Universidad de Zaragoza and IUMA, Spain).

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Poncelet–Darboux, Kippenhahn, and Szegő: projective geometry, matrices and orthogonal polynomials

Andrei Martinez-Finkelshtein

Baylor University

We study algebraic curves that are envelopes of families of polygons supported on the unit circle $\mathbb{T}$. We address, in particular, a characterization of such curves of minimal class and show that all realizations of these curves are essentially equivalent and can be described in terms of orthogonal polynomials on the unit circle (OPUC), also known as Szegő polynomials. These results have connections to classical results from algebraic and projective geometry, such as theorems of Poncelet, Darboux, and Kippenhahn; numerical ranges of a class of matrices; and Blaschke products and disk functions.

This is a joint work with Markus Hunziker, Taylor Poe, and Brian Simanek, all at Baylor University.

Bernstein type inequalities and potential theory

Béla Nagy

University of Szeged

Bernstein's inequality is well known in approximation theory. Although it was established more than 100 years ago, sharp forms for more general sets were found only 20 years ago. In this talk, we review asymptotically sharp Bernstein type inequalities and how the search for asymptotically sharp forms leads naturally to application of potential theory. In the end, some recent results will be presented.

The results are based on joint works with Vilmos Totik and Sergei Kalmykov.

Properties of level curves of a function generated by an Abel integral on three-sheeted torus

Semen Nasyrov

Kazan Federal University

We consider some Abel integral F on a three-sheeted torus T over the Riemann sphere $\mathbb{C}$ which is the Riemann surface of the function $w = \sqrt[3]{(z-a_1)(z-a_2)(z-a_3)}$; here a_k, $1 \le k \le 3$ are pairwise distinct points of $\mathbb{C}$. We describe the corresponding Abel integral on the universal covering of T and the differential-geometric structure of level lines of $\operatorname{Re} F$ which is a single-valued harmonic function. With the help of the study of the zero level line of the function, we can give a description of so-called Nuttall decomposition of T which plays an important role in the theory of Hermite–Padé diagonal approximations II.

If the points a_k are the vertices of some isosceles triangle, we can also completely describe the projection of the zero level line of $\operatorname{Re} F$ on $\mathbb{C}$. Our technique is based on the theory of the Weierstrass elliptic functions.

Multivariate mixed orthogonal functions satisfying three term relations

Teresa Pérez

Universidad de Granada

We study a non trivial extension of orthogonal functions introduced in [1] to several variables. This kind of functions satisfy mixed orthogonality conditions in the sense that the inner product of functions of different parity order is computed by means of a moment functional, and the inner product of elements of the same parity order is computed by a modification of the original moment functional. Existence conditions, three term relations with matrix coefficients, a Favard-type theorem for this kind of functions are proved. A method for constructing bivariate hybrid orthogonal functions from univariate orthogonal polynomials and univariate orthogonal functions is presented. Finally, we give a complete description of a sequence of mixed orthogonal functions on the unit ball on $\mathbb{R}^2$, that includes, as particular case, the classical orthogonal ball polynomials.

This is a joint work with Cleonice F. Bracciali, from Universidade Estadual Paulista, Brazil.

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Jackson-type estimates on the weighted unit ball

Miguel Piñar

Universidad de Granada

In this talk we explore the best approximation on the ball by means of orthogonal polynomials associated with weight functions that are invariant under reflection groups. A theory of orthogonal polynomials in this context can be developed in analogy to that for the orthogonal polynomials associated with standard spherical harmonics. Here, the standard first order partial differential operators are replaced by a family of commuting first order difference-differential operators: the so called Dunkl operators.

From Ginzburg–Landau vortices to Seiberg–Witten equations

Armen Sergeev

Steklov Mathematical Institute

Ginzburg–Landau vortices are the static solutions of Ginzburg–Landau equations arising in the superconductivity theory. They remind the hydrodynamical vortices which explains the origin of their name. If we switch on the time in the considered model then the vortices start to move and may collide with each other. For example, two vortices moving along the straight line towards each other scatter to the right angle. To describe the vortex dynamics it is convenient to use the so called adiabatic limit by tending their velocity to zero. The limiting behavior of vortex trajectories is described by geodesics on the vortex space in the metric determined by the kinetic energy of the model.

It turns out that this model has a nontrivial 4-dimensional analogue described by the Seiberg–Witten equations. These are equations on 4-dimensional Riemannian manifolds being, together with Yang–Mills equations, the limiting case of the supersymmetric Yang–Mills theory. We are most interested in the case of symplectic manifolds since such manifolds have, apart from Riemannian metric, also an almost complex structure compatible with this metric. We plug into the Seiberg–Witten equations a scale parameter and take the adiabatic limit by tending this parameter to infinity. The limiting trajectories are described by the pseudoholomorphic curves which may be considered as complex analogs of Ginzburg–Landau geodesics. Solutions of Seiberg–Witten equations in the adiabatic limit reduce to the families of Ginzburg–Landau vortices in the planes normal to the limiting pseudoholomorphic curve. In this sense the Seiberg–Witten equations may be considered as complex analogs of dynamical Ginzburg–Landau equations in which the role of "time" is played by the parameter running along the limiting pseudoholomorphic curve.

Recurrence Legendre polynomials

Vladimir Sorokin

Moscow State University

In this talk we discuss the recurrence Legendre polynomials, which are partly orthogonal with respect to the Lebesgue measure on the real segment symmetric about the unit circle. We show that the limiting distribution of their zeros is described in terms of a meromorphic function on a compact Riemann surface. This limiting measure can be interpreted in terms of the solution of a certain equilibrium problem for the logarithmic potential.

On functions of finite analytical complexity

Maria Stepanova

Moscow State University

An analytic function of two complex variables has finite complexity, if it is representable as a finite superposition of summation and analytic functions of one variable. The minimal possible depth of the superposition is called analytical complexity of the function. To the best of the speaker's knowledge, explicit examples of functions of complexity n, except for the cases $n = 0, 1, 2$ and infinity, were not known before. We will give examples of analytic functions and also polynomials of an arbitrary given finite complexity.

Bergman, Szegő, Faber and Christoffel: Boundary behavior

Nikos Stylianopoulos

University of Cyprus

Let G be a bounded simply-connected domain in the complex plane $\mathbb{C}$, with Jordan boundary Γ. We review some recent and present some new results on the asymptotic boundary behaviour of Bergman, Szegő and Faber polynomials, under various assumptions on the properties of Γ. These will lead to some new estimates for the asymptotic behaviour on Γ of the associated Christoffel functions.

Multiplicity of zeros of polynomials

Vilmos Totik

University of Szeged

In 1940 Paul Erdős and Paul Turán proved their basic discrepancy estimate stating that if a monic polynomial of degree n with real zeros has supremum norm $\|P_n\|_{[-1,1]} \le M_n 2^{-n}$ on $[-1, 1]$, then the normalized distribution of their zeros is closer than $C\sqrt{(\log M_n)/n}$ to the arcsine (Chebyshev) distribution. This implies that the highest multiplicity of the zeros is $\le C\sqrt{(\log M_n)n}$. In particular, if the norm is $O(1)2^{-n}$, then the highest multiplicity is $O(\sqrt{n})$, and Erdős and Turán conjectured that $\sqrt{n}$ order multiplicity can be achieved under this condition.

In this talk sharp bounds are given for the highest multiplicity of zeros of polynomials in terms of their norm on smooth Jordan curves and arcs. The results solve the just mentioned problem of Erdős and Turán. In addition, their bound on the multiplicities for the case $[-1, 1]$ will be made more precise by comparing the norm not to 2^{-n} but to the theoretical minimum 2^{1-n} (note that, by Chebyshev's theorem, $M_n \ge 2$, so from the Erdős–Turán estimate one cannot get a smaller bound for the multiplicity than $O(\sqrt{n})$, while if one can prove the bound $O(\sqrt{(\log N_n)n})$ from $\|P_n\|_{[-1,1]} \le N_n 2^{1-n}$, then this gives $o(\sqrt{n})$ as soon as $N_n = 1 + o(1)$). We shall also discuss why there are no such results on non-smooth curves and arcs, or on sets consisting of more than one component. The proofs use potential theory and a Faber-type modification of a result of G. Halász.

Real-valued semiclassical approximation for the asymptotics with complex-valued phases and its application to multiple orthogonal Hermite polynomials

Anna Tsvetkova

Ishlinsky Institute for Problems in Mechanics

The multiple orthogonal Hermite polynomials $H_{n_1,n_2}(z,a)$ are defined by the following recurrence relations (see [1]):

$$H_{n_1+1,n_2}(z,a) = (z+a)H_{n_1,n_2}(z,a) - \frac{1}{2}\left(n_1 H_{n_1-1,n_2}(z,a) + n_2 H_{n_1,n_2-1}(z,a)\right),$$
$$H_{n_1,n_2+1}(z,a) = (z-a)H_{n_1,n_2}(z,a) - \frac{1}{2}\left(n_1 H_{n_1-1,n_2}(z,a) + n_2 H_{n_1,n_2-1}(z,a)\right).$$

We construct the uniform Plancherel–Rotach-type asymptotics of diagonal polynomials $H_{n,n}(z,a)$ as $n \to \infty$. To obtain the result we develop the method which we call "real-valued semiclassical approximation for the asymptotics with complex-valued phases" (another approach based on the construction of decompositions of bases of difference equations was recently developed by A. I. Aptekarev and D. N. Tulyakov).

The discussed approach can be applied to construct asymptotics for a wide class of orthogonal polynomials. The idea of the method is to introduce a small artificial parameter $h \sim \frac{1}{n}$ and a continuous function $\varphi(x,z,a)$ such that $H_{n,n}(z,a) = \varphi(nh,z,a)$. This allows us to reduce the system that defines the polynomials to a pseudo-differential equation for φ, where x is a variable and (z,a) are parameters. Seeking its solution in the WKB-form, one obtains the Hamiltonian-Jacobi equation with the complex-valued Hamiltonian connected with the third-order algebraic curve. In general case, to obtain the result for such problems, a transition from real variable x to a complex one is made. In this problem, we propose a different approach based on a reduction of the original problem to three equations, two of which have asymptotics with purely imaginary phases, and the symbol of the third one has the form $\cos p + V_0(x) + hV_1(x) + O(h^2)$. The construction of an asymptotic solution of the last equation is described in [2], what allows us to obtain a uniform asymptotics for $H_{n,n}(z,a)$ in the form of the Airy function Ai of the real-valued argument.

The talk is based on the joint work with A. I. Aptekarev (Keldysh Institute of Applied Mathematics RAS), S. Yu. Dobrokhotov (Ishlinsky Institute for Problems in Mechanics RAS) and D. N. Tulyakov (Keldysh Institute of Applied Mathematics RAS).

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Polynomial solutions of linear differential operators and Bochner's theorem

Mikhail Tyaglov

Shanghai Jiao Tong University

Consider linear differential operators of the form

$$\mathcal{L}_r u \stackrel{def}{=} \sum_{j=0}^{r} Q_j(z) \frac{d^j u(z)}{dz^j},$$

where $\deg Q_j = n_j$, $j = 0, 1, \ldots, r$, and $Q_0(0) = 0$.

In the talk, we discuss operators $\mathcal{L}_r$ having infinite sequences of polynomials eigenfunctions. We state necessary and sufficient conditions for such operators to have a complete sequences of eigenpolynomials and describe cases when those conditions fail. In particular, we found all the operators $\mathcal{L}_2$ with infinite sequences of polynomial eigenfunctions, including the cases missed by S. Bochner [1], and give examples of eigenpolynomial sequences for $\mathcal{L}_3$ and $\mathcal{L}_4$.

This is a joint work with Alexander Dyachenko.

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Tensors, polynomials and rank increment conjecture

Eugene Tyrtyshnikov

Institute of Numerical Mathematics

We discuss some known and unknown properties of the tensor rank and especially the rank increment conjecture and some related results.

Riesz energy problems in unbounded sets of $\mathbb{R}^d$

Frank Wielonsky

Université Aix-Marseille

We investigate Riesz energy problems on unbounded conductors in $\mathbb{R}^d$ in the presence of general external fields Q. We provide new sufficient conditions on Q for the existence of an equilibrium measure and the compactness of its support. Particular attention is paid to the case of the hyperplanar conductor $\mathbb{R}^d$, embedded in $\mathbb{R}^{d+1}$, when the external field is created by the potential of a simple discrete measure ν outside of $\mathbb{R}^d$. An extension of a classical theorem by de La Vallée-Poussin is established which may be of independent interest.

This is a joint work with P. Dragnev, R. Orive, and E. B. Saff.

On $L^2_{\mathbb{R}}$-best rational approximants to Markov functions on several intervals

Maxim Yattselev

Indiana University–Purdue University Indianapolis

Let $f(z) = \int (z-x)^{-1} d\mu(x)$, where μ is a Borel measure supported on several subintervals of $(-1,1)$ with smooth Radon–Nikodym derivative. In this talk strong asymptotic behavior of the error of approximation $(f - r_n)(z)$ will be described, where $r_n(z)$ is the $L^2_{\mathbb{R}}$-best rational approximant to $f(z)$ on the unit circle with n poles inside the unit disk.

Pointwise Remez inequality

Peter Yuditskii

Johannes Kepler University Linz

The classical Remez inequality provides an exact estimate for a polynomial on a given interval if it is known that the polynomial is bounded by one on a subset of this interval of the given Lebesgue measure. To be precise, let Π_n denote the set of polynomials of degree at most n. For a subset E of the interval $[-1,1]$ let

$$\Pi_n(E) := \{P_n \in \Pi_n : \ |P_n(x)| \le 1, \ x \in E\}.$$

The Lebesgue measure of E is denoted by $|E|$. For $\delta \in (0,1)$ we define

$$M_{n,\delta} = \sup_{E:|E|=2-2\delta} \ \sup_{P_n \in \Pi_n(E)} \ \sup_{x\in[-1,1]} |P_n(x)|.$$

According to Remez $M_{n,\delta} = T_n\left(\frac{1+\delta}{1-\delta}\right)$, where $T_n(z)$ is the classical Chebyshev polynomial, $T_n(z) = \frac{1}{2}(\zeta^n + \zeta^{-n})$, $z = \frac{1}{2}(\zeta + \zeta^{-1})$.

Since the mid-90s, based on the previous results of T. Erdélyi, E. B. Saff and himself, at several international conferences Vladimir Andrievskii raised the following problem. Find

$$L_{n,\delta}(x_0) = \sup_{E:|E|=2-2\delta} \ \sup_{P_n \in \Pi_n(E)} |P_n(x_0)|, \quad x_0 \in [-1,1].$$

Note that $\sup_{x_0\in[-1,1]} L_{n,\delta}(x_0) = M_{n,\delta}$.

In the talk we present a solution of Andievskii's problem on the *pointwise* Remez inequality.